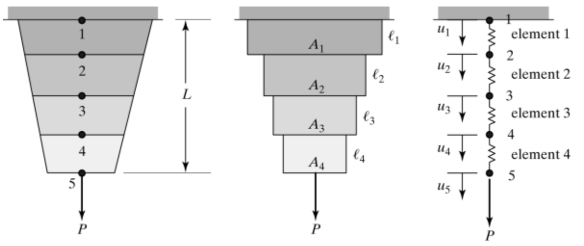
Chap. 1 Introduction to FE Analysis



CAE (Computer Aided Engineering)

❑ CAE 정의

- 실생활에서 발생되는 여러 가지 물리적 현상을 예측하기 위해
- 컴퓨터 수치해석 기법을 통해 그 결과를 미리 Simulation하는 기법

❑ CAE 해석기법 종류

- 유한요소법 (Finite Element Method; FEM)
- 유한차분법 (Finite Difference Method; FDM)
- 경계요소법 (Boundary Element Method: BEM)
- 유한체적법 (Finite Volume Method; FVM)

❑ CAE 적용 분야

- 구조해석, 충돌해석, 열/유체 유동해석, 소음/진동 해석
- 각종 성형공정 해석(단조, 압출, 박판성형, 사출성형, 주조 등)
- 전자기장 해석, 광학계 해석, 분자구조 해석 등

유한요소법 (Finite Element Method; FEM)

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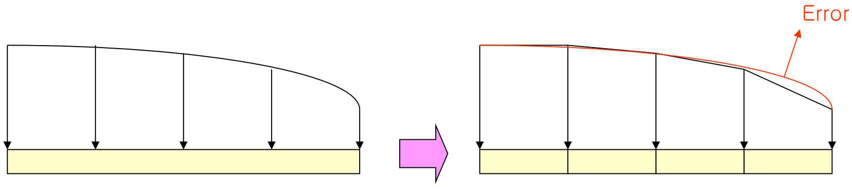
❑ 유한요소법 특징

- 지배방정식 (미분방정식) ➲ 적분 형태의 방정식 (variational principle)
- 연속체(continuum)로 정의된 전체 영역을 이산화(discretization)
  : 다수개의 유한요소(finite element)로 구성 ➲ Mesh
- 각 요소 내에서 보간함수(interpolation function)를 사용하여 실제 결과를 근사화



실제 온도분포 (연속체)　　　유한요소 근사화 결과 (선형보간)

Engineering Problems

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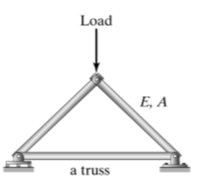
❑ Solid Mechanics Problem

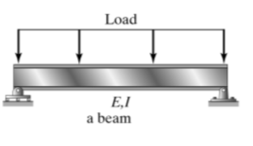
A truss system

- Parameters:
- Load:



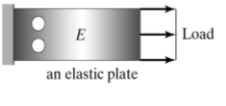
An elastic beam

- Parameters:
- Load:



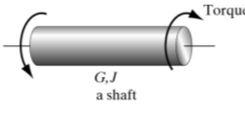
An elastic plate

- Parameters:
- Load:



An elastic shaft

- Parameters:
- Load:



Engineering Problems

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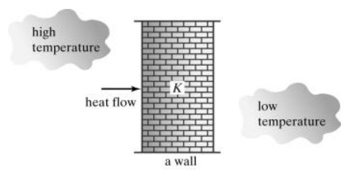
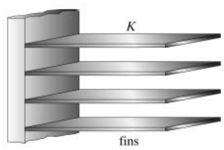
❑ Heat Transfer Problem

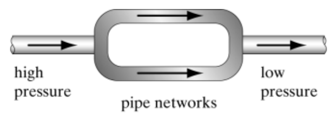
- Parameter:
- Loads:

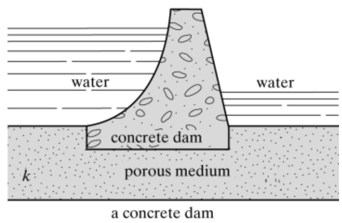




❑ Fluid Flow Problem

- Parameter:
- Loads:





Engineering Problems – Eq'n & Solution

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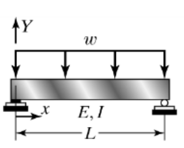
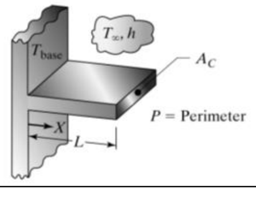
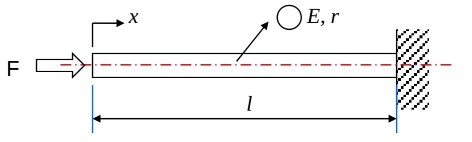
Problem Type		Solution
Solid mechanics 	G.E. $EI\dfrac{d^2Y}{dX^2}=\dfrac{wX(L-X)}{2}$ B.C. $X=0$ 에서 $Y=0$ $X=L$ 에서 $Y=0$	거리 X의 함수에 의한 보(beam)의 처짐 Y $Y=\dfrac{w}{24EI}\left(-X^4+2LX^3-L^3X\right)$
Heat transfer 	G.E. $\dfrac{d^2T}{dX^2}-\dfrac{hp}{kA_C}(T-T_\infty)=0$ B.C. $X=0$ 에서 $T=T_{base}$ $L\to\infty$ 일때 $T=T_\infty$	함수 X에 의한 핀(fin)에 따른 온도 분포 $T=T_\infty+(T_{base}-T_\infty)e^{-\sqrt{\frac{hp}{kA_C}}X}$

❑ Ex) Simple compression of a rod

- 소재: 원형 봉재 (rod)
- 하중: 축방향 하중
- 재료의 위치(x)에 따른 변위(u)는?



❑ 기본 가정

- 재료의 단면은 일정함 (radius: r)
- 하중은 봉의 중심에 축방향으로 작용
- 재료는 선형 탄성재료로 가정
- 주변 온도는 균일하다고 가정 (iso-thermal)
- 등방성(isotropic) 재료로 가정
- 균일한(homogeneous) 재료로 가정

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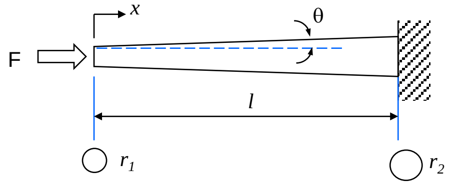
❑ Ex 2) Simple compression of a rod

- 소재의 단면이 변하는 경우
- 하중: 축방향 하중
- 재료의 위치(x)에 따른 변위(u)는?



❑ 기본 가정

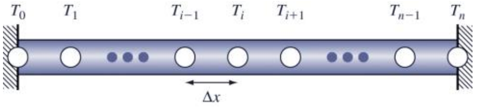
- ~~재료의 단면은 일정함 (radius: r)~~
- 하중은 봉의 중심에 축방향으로 작용
- 재료는 선형 탄성재료로 가정
- 주변 온도는 균일하다고 가정 (iso-thermal)
- 등방성(isotropic) 재료로 가정
- 균일한(homogeneous) 재료로 가정

❑ Requirements

- There are many practical engineering problems we cannot obtain exact solutions
- Difficulties in obtaining the exact solutions (analytical solutions)
  - The complex nature of governing equations (differential equations)
  - Difficulties that arise from dealing with the boundary and initial conditions

❑ Numerical Method

- To obtain numerically-approximated solutions only at discrete points (nodes)
- Needs discretization of the analysis domain (mesh generation)
- Two popular numerical methods
  - Finite difference method (FDM)
  - Finite element method (FEM)



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❑ Finite Difference Method (FDM)

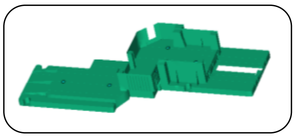
- The differential equation is written for each node, and the derivatives are replaced by difference equations (resulting in a set of simultaneous equation)
- Easy to understand and to employ in simple problems
- Difficult to apply to problems with complex geometries or boundary conditions

❑ Finite Element Method (FEM)

- Uses integral formulations to create a system of algebraic equations
- An approximate continuous function is assumed to represent the solution for each element
- The complete solution is then generated by connecting or assembling the individual solutions, allowing for continuity at the inter-elemental boundaries

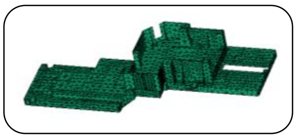
Basic Steps in CAE Analysis

Pre-processing



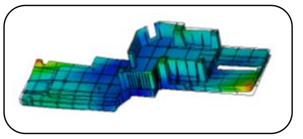
- Geometry modeling (3D CAD)
- Mesh generation (element type selection)
- Boundary condition imposition
- Material property

FEM solving



- Analysis type selection (static/ dynamic, steady/non-steady..)
- Solver type selection
- Solving parameter (time step, convergence criterion…)

Post-processing



- Result display (stress, force, displacement, temperature…)
- Evaluation of the results

Basic Steps in Finite Element Method

Step I. Preprocessing Phase

1. Create and discretize the solution domain into finite elements (subdivide the problem into nodes and elements)
2. Assume a shape function to represent the physical behavior of an element
3. Develop equations for an element (construct the elementary stiffness matrix)
4. Assembly the elements to represent the entire problem (construct the global stiffness matrix)
5. Apply boundary conditions, initial conditions, and loading

Step II. Solution Phase

1. Solve a set of linear or nonlinear algebraic equations simultaneously to obtain nodal results (DOF solutions: nodal displacements, temperatures, etc.)

Step III. Postprocessing Phase

1. Obtain other important information (Ex. Stresses, heat fluxes, etc.)

Finite Element Formulations

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❑ Direct Formulation

- Follows the basic FEM steps (steps 1 ~ 7) as a direct manner
- Ease to understand the basic FEM steps (not used in real FE formulation)

❑ Minimum Total Potential Energy Formulation

- A common approach in generating FE models in solid mechanics
- Based on the strain energy during the deformation by the external forces

❑ Weighted Residual Formulation

- Based on assuming an approximate solution for the governing differential equation
- The assumed solution leads to some errors (residuals)
- To make these residuals vanish over some selected intervals or at some points
- Collocation method, Subdomain method, Galerkin method, Least-square method)

Example: A bar with a variable cross section

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Equilibrium equation in the y-direction : $P-(\sigma_{avg})A(y)=0$ … (1)

Hooke's law : $\sigma = E\varepsilon$ … (2) Strain : $\varepsilon = \frac{du}{dy}$ … (3)

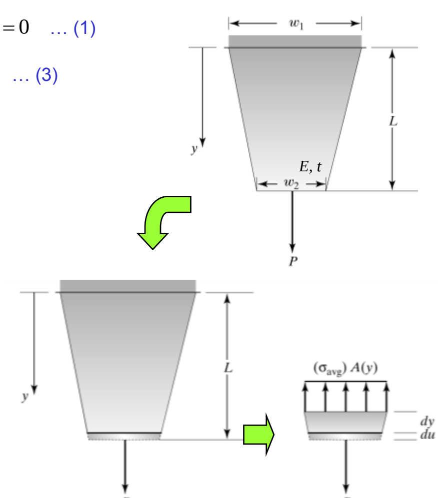
Substituting eqns (2) & (3) into (1) :

… (4)

By using linear interpolation:

Integrating (4) with (5) $u(y)=\int_0^y \frac{Pdy}{EA(y)} = \int_0^y \frac{Pdy}{E\left[w_1+\left(\frac{w_2-w_1}{L}\right)y\right]t}$

Exact solution



Direct Formulation (Preprocessing Phase)

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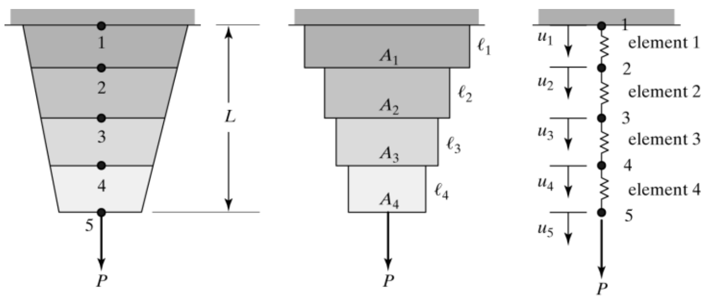
Step 1. Discretize the solution domain into finite elements



- The given bar is modeled using four individual segments (5 nodes and 4 elements)
- The cross-sectional area of each element is represented by an averaging value (A_i)

Direct Formulation (Preprocessing Phase)

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Step 2. Assume a solution that approximates the behavior of an element

$\sigma = F/A \quad \varepsilon = \Delta\ell/\ell \quad \sigma = E\varepsilon$ ➡

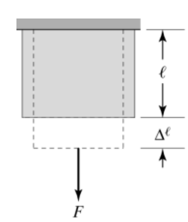
Considering a linear spring eq'n: $F = Kx$ ➡

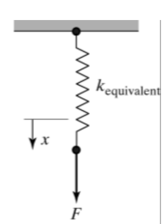
The bar can be represented by a model consisting of 4 springs (element)

➡

where $k_{eq} = \dfrac{(A_{i+1} - A_i)E}{2l}$: the equivalent element stiffness

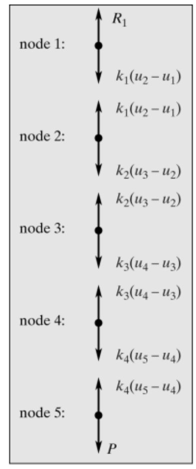
A_i & A_{i+1} : the cross-sectional areas of the member at node i & $i+1$





Direct Formulation (Preprocessing Phase)

Step 2. Assume a solution that approximates the behavior of an element



Static equilibrium equations:

Node 1:

Node 2:

Node 3:

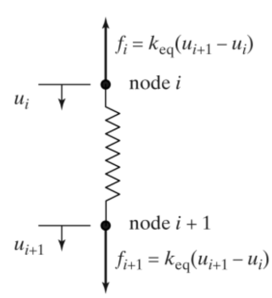
Node 4:

Node 5:

Direct Formulation (Preprocessing Phase)

Step 3. Develop equations for an element

Considering the equilibrium equation for an element:



where f_i and f_{i+1} : the internally transmitted forces at i *and* $i+1$

u_i and u_{i+1} : the end displacements at i *and* $i+1$

In a matrix form,

Direct Formulation (Preprocessing Phase)

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Step 4. Assemble all elements to present the entire problem

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[\mathbf{K}]^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$[\mathbf{K}]^{(1G)} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix}$$

$$\begin{Bmatrix} f_4 \\ f_5 \end{Bmatrix} = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix}$$

$$[\mathbf{K}]^{(4)} = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}$$

$$[\mathbf{K}]^{(4G)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix}$$

Direct Formulation (Preprocessing Phase)

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Step 4. Assemble all elements to present the entire problem

$$[\mathbf{K}]^{(G)} = [\mathbf{K}]^{(1G)} + [\mathbf{K}]^{(2G)} + [\mathbf{K}]^{(3G)} + [\mathbf{K}]^{(4G)}$$

: Global stiffness matrix

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix}$$

Direct Formulation (Preprocessing Phase)

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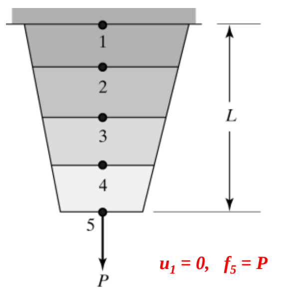
Step 5. Apply boundary conditions and loads

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix}$$



$$[K]\{u\} = \{f\}$$

[K]: Stiffness matrix
{f}: load vector
{u}: solution vector

Direct Formulation (Solution Phase)

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Step 6. Solve a system of algebraic equations (p.14~16)

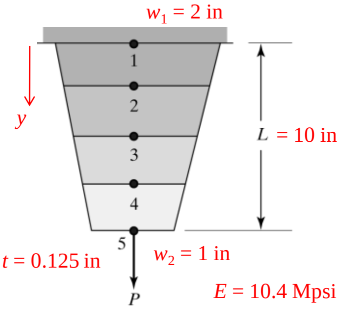
Cross-sectional area of the bar at each node

$$A(y) = \left\{ w_1 + \left(\frac{w_1 - w_2}{L} \right) y \right\} t = 0.25 - 0.0125y$$

Equivalent stiffness coefficient for each element

$$k_i = \frac{(A_{i+1} - A_i)E}{2l} \qquad [\mathbf{K}]^{(i)} = \begin{bmatrix} k_i & -k_i \\ -k_i & k_i \end{bmatrix}$$

Assembly the global stiffness matrix and solve

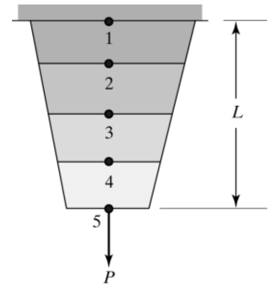


$$10^3 \begin{bmatrix} 1820 & -845 & 0 & 0 \\ -845 & 1560 & -715 & 0 \\ 0 & -715 & 1300 & -585 \\ 0 & 0 & -585 & 585 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 10^3 \end{Bmatrix}$$

Direct Formulation (Postprocessing Phase)

Step 7. Obtain other engineering information (p.16~18)

Average normal stresses in each element



Reaction forces at each node

$$R_1 = k_1(u_2 - u_1) = 975\times10^3(0.001026-0) = 1000lb$$

Reaction forces at each node

$$\{\mathbf{R}\} = [\mathbf{K}]\{\mathbf{u}\} - \{\mathbf{F}\} = \begin{Bmatrix} -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

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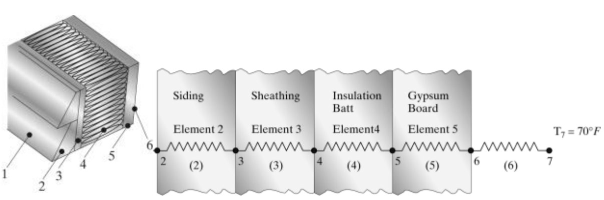
Direct Formulation (Other Examples)



Example 1-2

Figure 1-7
Finite element model of Example 1.2.

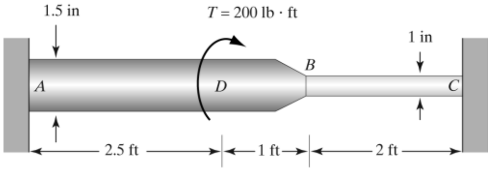


Figure 1-12
A schematic of the shaft in Example 1.3.

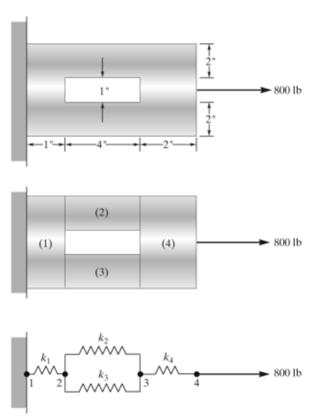
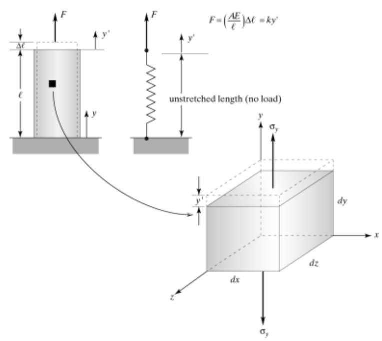


Figure 1-13
A schematic of the steel plate in Example 1.4.

❑ Minimum Total Potential Energy Formulation

- During the deformation, the work done by external forces is stored in the form of elastic energy, called strain energy
- Minimum potential energy: In a stable system, the total potential energy in the system is minimized in the case of the equilibrium state
- Strain energy under the deformation of dy



- Strain energy in terms of stress(σ) and strain(ε)

Minimum Total Potential Energy Formulation
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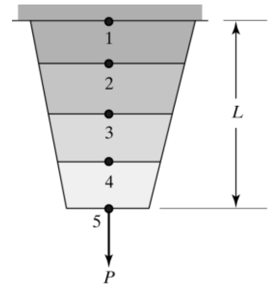
❑ Minimum Total Potential Energy Formulation (cont'd)

- Strain energy for an element under axial loading
- The total potential energy(Π) for a body consisting of n elements and m nodes

  : defined by the difference between the total potential energy and the work done by the external forces
- The minimum total potential energy principle : The 1st derivative becomes zero

Minimum Total Potential Energy Formulation

❑ Example: A bar with a variable cross section



- The strain energy for an arbitrary element (e)
- The axial strain
- Minimizing the strain energy w.r.t. u_i and u_{i+1}

Minimum Total Potential Energy Formulation

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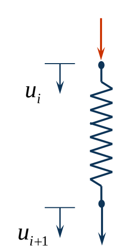
❑ Example: A bar with a variable cross section (cont'd)



- Minimizing the work done by the external force at nodes i and $i+1$

$$\frac{\partial}{\partial u_i}\left(F_i u_i\right) = F_i$$

$$\frac{\partial}{\partial u_{i+1}}\left(F_{i+1} u_{i+1}\right) = F_{i+1}$$

- The minimum total potential energy formulation then leads to the same matrix equation consisting of a global stiffness matrix and a load vector

$$\frac{\partial \Pi}{\partial u_i} = \frac{\partial}{\partial u_i}\sum_{e=1}^{n}\Lambda^{(e)} - \frac{\partial}{\partial u_i}\sum_{i=1}^{m}F_i u_i = 0$$

Weighted Residual Formulations

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❑ Weighted Residual Formulation

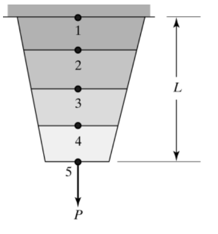
- Based on assuming an approximate solution for the governing differential equation
- The assumed solution leads to some errors (residuals)
- To make these residuals vanish over some selected intervals or at some points

❑ Example: A bar with a variable cross section

- The governing differential equation and the corresponding boundary condition

$$A(y)E\frac{du}{dy} - P = 0 \qquad \text{[B.C.]} \quad u(0) = 0$$

- Approximate solution (to meet B.C.) $u(y) = c_1 y + c_2 y^2 + c_3 y^3$



⇨

⇨ $\Re / E = (0.25 - 0.0125y)(c_1 + 2c_2 y + 3c_3 y^2) - 96.154 \times 10^{-6}$

Weighted Residual Formulations

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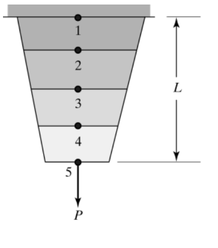
❑ Collocation Method (배열법)

- The residual is forced to be zero at as many points as the number of coefficients
- Then solve a system of linear equation to obtain coefficients
- Example: A bar with a variable cross section

  : The error (residuals) function becomes zero at three points: y = L/3, 2L/3, L

$$\Re / E = (0.25 - 0.0125y)(c_1 + 2c_2 y + 3c_3 y^2) - 96.154 \times 10^{-6}$$



⇨

⇨ $u(y) = 423.0776 \times 10^{-6} y + 21.65 \times 10^{-15} y^2 + 1.153848 \times 10^{-6} y^3$

Weighted Residual Formulations

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❑ Subdomain Method (부영역법)

- The integral of the error function over some selected subintervals is forced to be 0
- The number of subintervals must equal the number of unknown coefficients
- Example: A bar with a variable cross section

  : The integral becomes zero at three subintervals: [0, L/3], [L/3, 2L/3], [2L/3, L]

$$\Re / E = (0.25 - 0.0125y)(c_1 + 2c_2 y + 3c_3 y^2) - 96.154\times10^{-6}$$

⇨

⇨ $u(y) = 391.35088\times10^{-6}y + 6.075\times10^{-6}y^2 + 809.61092\times10^{-9}y^3$

Weighted Residual Formulations

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❑ Galerkin Method

- Requires the error (residual) to be orthogonal to some weighting functions Φ_i
- The weight functions are chosen to be members of the approximate solution
- Example: A bar with a variable cross section (Φ_1: y, Φ_2: y^2, Φ_3: y^3)

$$\Re / E = (0.25 - 0.0125y)(c_1 + 2c_2 y + 3c_3 y^2) - 96.154\times10^{-6}$$

⇨

⇨ $u(y) = 400.642\times10^{-6}y + 4.006\times10^{-6}y^2 + 0.935\times10^{-6}y^3$

Weighted Residual Formulations

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❑ Least-Square Method (최소제곱법)

- Requires the error (residual) to be minimized wrt. the unknown coefficients

- Example: A bar with a variable cross section

$$\Re / E = (0.25 - 0.0125y)\left(c_1 + 2c_2 y + 3c_3 y^2\right) - 96.154\times10^{-6}$$

⇨

⇨ $u(y) = 389.793\times10^{-6}\, y + 6.442\times10^{-6}\, y^2 + 0.789\times10^{-6}\, y^3$

Weighted Residual Formulations

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❑ Comparison of Weighted Residual Solutions

TABLE 1.6 Comparison of weighted residual results

Location of a Point Along the Bar (in)	Displacement Results from the Exact Solution Eq. (1.53) (in)	Displacement Results from the Collocation Method Eq. (1.57) (in)	Displacement Results from the Subdomain Method Eq. (1.59) (in)	Displacement Results from the Galerkin Method Eq. (1.62) (in)	Displacement Results from the Least-Squares Method Eq. (1.64) (in)
$y = 0$	0	0	0	0	0
$y = 2.5$	0.001027	0.001076	0.001029	0.001041	0.001027
$y = 5.0$	0.002213	0.002259	0.002209	0.002220	0.002208
$y = 7.5$	0.003615	0.003660	0.003618	0.003624	0.003618
$y = 10$	0.005333	0.005384	0.005330	0.005342	0.005331