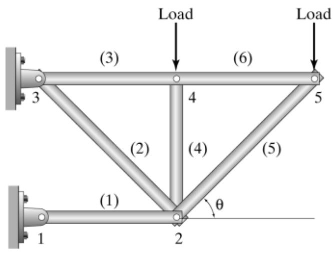
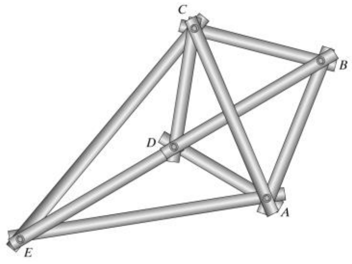
Chap. 2 FE Formulation for Trusses





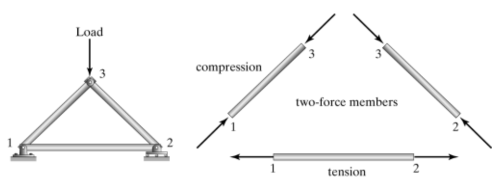
Characteristics of a Truss

❑ Definition

- An engineering structure consisting of straight members at their ends by means of bolts, rivets, pins, or welding.
- Can be found in many engineering structures: power transmission towers, bridges, roofs of buildings, etc.

❑ Basic Assumptions in Analysis

- The truss members are connected together by smooth pins (2d) or a ball-and-socket joint (3d)
- All loads are applied at the joint (axial)
- The weight of members are negligible



Statically Determinate vs. Indeterminate

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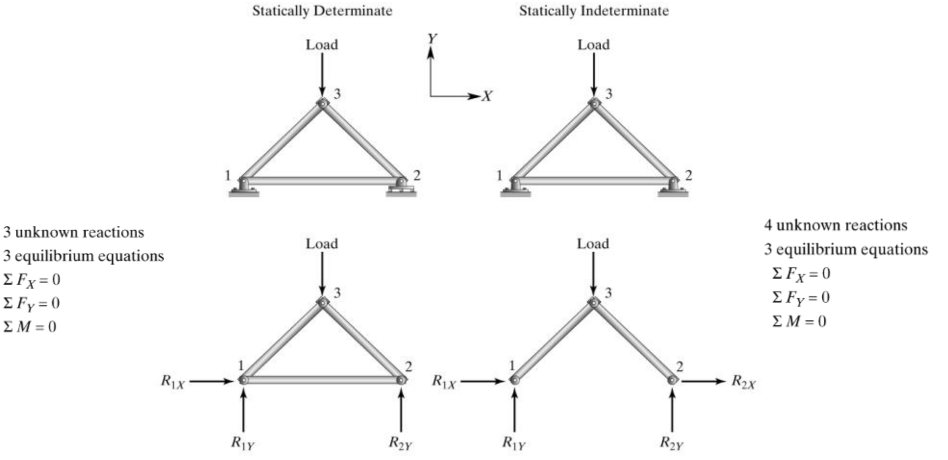

Finite Element Formulation

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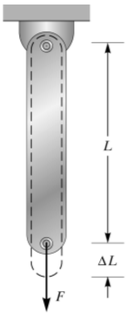
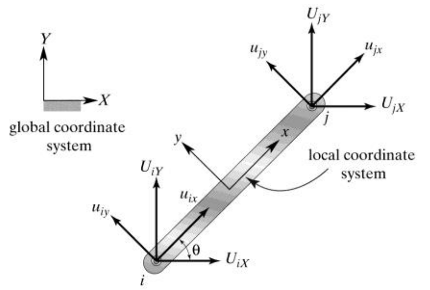
❑ Displacements of a Single Member

- The average stress in a member: (1)
- The average strain of the member: (2)
- Hooke's law: (3)
- Combining eqs. (1) ~ (3) (4)
- Linear spring eqn with an equivalent stiffness

(5)

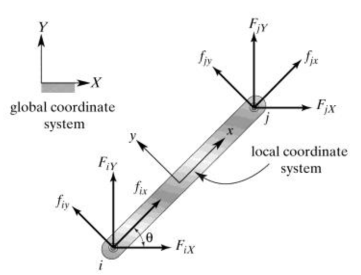


❑ Displacements: Global vs. Local



$$\{\mathbf{U}\} = [\mathbf{T}]\{\mathbf{u}\}$$

❑ Forces: Global vs. Local



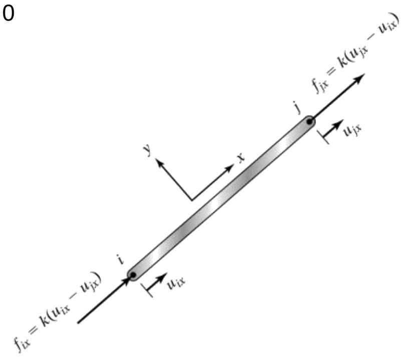
$$\{\mathbf{F}\} = [\mathbf{T}]\{\mathbf{f}\}$$

❑ Relations between Forces and Displacements

- Trusses: Only axial forces & displacements
- Forces & displacements in the local y-direction: 0

$$f_{ix} = k\left(u_{ix} - u_{jx}\right) \qquad f_{iy} = 0$$

$$f_{jx} = k\left(u_{jx} - u_{ix}\right) \qquad f_{jy} = 0$$



$$k = k_{eq} = \frac{AE}{L}$$

❑ Elementary Stiffness Matrix

$$\{\mathbf{f}\} = [\mathbf{K}]\{\mathbf{u}\}$$

$$[\mathbf{T}]^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Elementary stiffness matrix : $[K]^e$

Example: A Balcony Truss Problem

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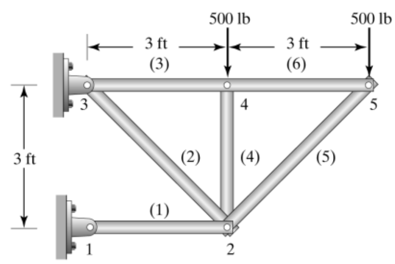
Inputs

- Material : Douglas-fir wood
- Modulus of elasticity (E) : 1.90 x 10^6 psi
- Cross-sectional area (A) : 8 in^2

Outputs

- Deflection of each joint
- Average stress in each member

Analytic solution : Section 2.6 (pp. 110 ~ 111)



Example: A Balcony Truss Problem

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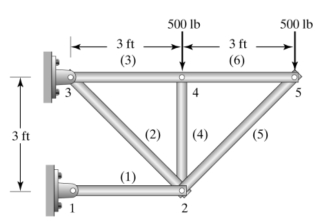
Step 1. Discretize the solution domain into finite elements



- Element: Each truss member
- Node: Each joint connecting members
- Discretized into 5 nodes and 6 elements

Element	Node i	Node j	Angle (deg)	Length (in)
(1)	1	2	0	36.0
(2)	2	3	135	50.9
(3)	3	4	0	36.0
(4)	2	4	90	36.0
(5)	2	5	45	50.9
(6)	4	5	0	36.0

Example: A Balcony Truss Problem

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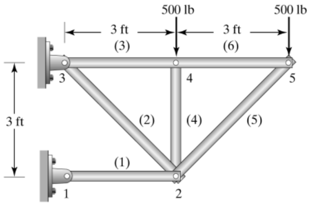
Step 2. Assume a solution that approximates the behavior of an element

- For elements (1), (3), (4), and (6)

$$k = \frac{AE}{L} = \frac{(8\,\text{in}^2)\left(1.90\times10^6\ \frac{\text{lb}}{\text{in}^2}\right)}{36\,\text{in}} = 4.22\times10^5\ \frac{\text{lb}}{\text{in}}$$

- For elements (2) and (5)

$$k = \frac{AE}{L} = \frac{(8\,\text{in}^2)\left(1.90\times10^6\ \frac{\text{lb}}{\text{in}^2}\right)}{50.9\,\text{in}} = 2.98\times10^5\ \frac{\text{lb}}{\text{in}}$$



Example: A Balcony Truss Problem

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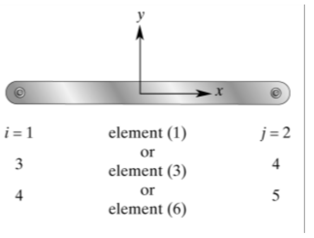
Step 3. Develop equations for an element

$$[K]^e = k\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

- For elements (1), (3), and (6): $\theta = 0$

$$[K]^{(e)} = 4.22\times10^5\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

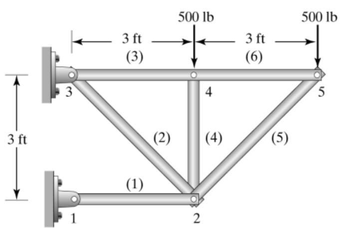


Example: A Balcony Truss Problem

Step 3. Develop equations for an element

- For element (4): θ = 90

$$[K]^{(4)} = 4.22\times10^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} U_{2X} \\ U_{2Y} \\ U_{4X} \\ U_{4Y} \end{matrix}$$



- For element (2): θ = 135

$$[K]^{(2)} = 2.98\times10^5 \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{matrix} U_{2X} \\ U_{2Y} \\ U_{3X} \\ U_{3Y} \end{matrix}$$

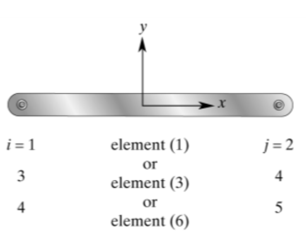
- For element (5): θ = 45

$$[K]^{(5)} = 2.98\times10^5 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{matrix} U_{2X} \\ U_{2Y} \\ U_{5X} \\ U_{5Y} \end{matrix}$$

Example: A Balcony Truss Problem

Step 4. Assemble all elements to present the entire problem

$$[K]^{(1)} = 4.22\times10^5 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} U_{1X} \\ U_{1Y} \\ U_{2X} \\ U_{2Y} \end{matrix}$$



Example: A Balcony Truss Problem

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Step 4. Assemble all elements to present the entire problem

$$[\mathbf{K}]^{(G)} = 10^5 \begin{bmatrix} 4.22 & 0 & -4.22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4.22 & 0 & 7.2 & 0 & -1.49 & 1.49 & 0 & 0 & -1.49 & -1.49 \\ 0 & 0 & 0 & 7.2 & 1.49 & -1.49 & 0 & -4.22 & -1.49 & -1.49 \\ 0 & 0 & -1.49 & 1.49 & 5.71 & -1.49 & -4.22 & 0 & 0 & 0 \\ 0 & 0 & 1.49 & -1.49 & -1.49 & 1.49 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.22 & 0 & 8.44 & 0 & -4.22 & 0 \\ 0 & 0 & 0 & -4.22 & 0 & 0 & 0 & 4.22 & 0 & 0 \\ 0 & 0 & -1.49 & -1.49 & 0 & 0 & -4.22 & 0 & 5.71 & 1.49 \\ 0 & 0 & -1.49 & -1.49 & 0 & 0 & 0 & 0 & 1.49 & 1.49 \end{bmatrix} \begin{matrix} U_{1X} \\ U_{1Y} \\ U_{2X} \\ U_{2Y} \\ U_{3X} \\ U_{3Y} \\ U_{4X} \\ U_{4Y} \\ U_{5X} \\ U_{5Y} \end{matrix}$$

Example: A Balcony Truss Problem

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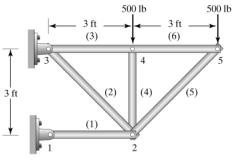
Step 5. Apply boundary conditions and loads



- Displacement BC: θ = Nodes 1 & 3 are fixed
- Force BS: 500lb are applied at nodes 4 & 5

$$10^5 \begin{bmatrix} 4.22 & 0 & -4.22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4.22 & 0 & 7.2 & 0 & -1.49 & 1.49 & 0 & 0 & -1.49 & -1.49 \\ 0 & 0 & 0 & 7.2 & 1.49 & -1.49 & 0 & -4.22 & -1.49 & -1.49 \\ 0 & 0 & -1.49 & 1.49 & 5.71 & -1.49 & -4.22 & 0 & 0 & 0 \\ 0 & 0 & 1.49 & -1.49 & -1.49 & 1.49 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.22 & 0 & 8.44 & 0 & -4.22 & 0 \\ 0 & 0 & 0 & -4.22 & 0 & 0 & 0 & 4.22 & 0 & 0 \\ 0 & 0 & -1.49 & -1.49 & 0 & 0 & -4.22 & 0 & 5.71 & 1.49 \\ 0 & 0 & -1.49 & -1.49 & 0 & 0 & 0 & 0 & 1.49 & 1.49 \end{bmatrix} \begin{Bmatrix} U_{1X} \\ U_{1Y} \\ U_{2X} \\ U_{2Y} \\ U_{3X} \\ U_{3Y} \\ U_{4X} \\ U_{4Y} \\ U_{5X} \\ U_{5Y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -500 \\ 0 \\ -500 \end{Bmatrix}$$

Example: A Balcony Truss Problem

Step 6. Solve a system of algebraic equations

- BC imposition : 10 x 10 => 6 x 6

$$10^5\begin{bmatrix} 7.2 & 0 & 0 & 0 & -1.49 & -1.49 \\ 0 & 7.2 & 0 & -4.22 & -1.49 & -1.49 \\ 0 & 0 & 8.44 & 0 & -4.22 & 0 \\ 0 & -4.22 & 0 & 4.22 & 0 & 0 \\ -1.49 & -1.49 & -4.22 & 0 & 5.71 & 1.49 \\ -1.49 & -1.49 & 0 & 0 & 1.49 & 1.49 \end{bmatrix}\begin{Bmatrix} U_{2X} \\ U_{2Y} \\ U_{4X} \\ U_{4Y} \\ U_{5X} \\ U_{5Y} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -500 \\ 0 \\ -500 \end{Bmatrix} \Rightarrow \begin{Bmatrix} U_{1X} \\ U_{1Y} \\ U_{2X} \\ U_{2Y} \\ U_{3X} \\ U_{3Y} \\ U_{4X} \\ U_{4Y} \\ U_{5X} \\ U_{5Y} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ -0.00355 \\ -0.01026 \\ 0 \\ 0 \\ 0.00118 \\ -0.0114 \\ 0.00240 \\ -0.0195 \end{Bmatrix}$$

Example: A Balcony Truss Problem

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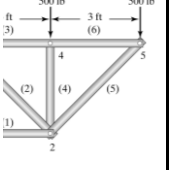
Step 7. Obtain other engineering information



- Reaction forces :

$$\begin{Bmatrix} R_{1X} \\ R_{1Y} \\ R_{2X} \\ R_{2Y} \\ R_{3X} \\ R_{3Y} \\ R_{4X} \\ R_{4Y} \\ R_{5X} \\ R_{5Y} \end{Bmatrix}=\begin{Bmatrix} 1500 \\ 0 \\ 0 \\ 0 \\ -1500 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

- Internal force and stress (for element 5)

$$f_{\text{int}} = k(u_{jx} - u_{ix}) = \frac{AE}{L}(u_{5x} - u_{2x}) = -696lb \;\; \Rightarrow \therefore \sigma = \frac{f_{\text{int}}}{A} = -87\,psi$$