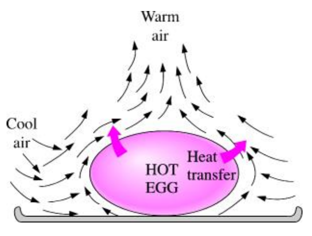
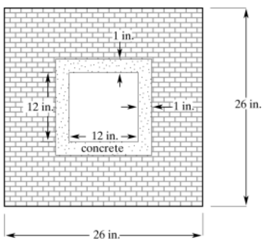
Chap. 6 Analysis of 2d Heat Transfer Problem

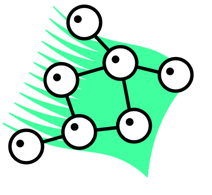




[Review] 열전달의 기본 메커니즘

❑ 전도 (Conduction)

- 입자들의 상호 작용에 의해 보다 활동적인 입자로부터 주변의 덜 활동적인 분자로 에너지가 전달되는 현상
- 고체, 액체, 기체에서 모두 발생 (고체의 경우가 가장 지배적임)
- 고체: 격자에서의 분자 진동 및 자유전자의 에너지 이송에 의해 발생
- 액체 및 기체: 분자의 불규칙적인 운동에 의한 충돌(collision) 및 확산(diffusion)에 의해 발생
- 예) 컵에 뜨거운 물이 담겨있을 때 컵 표면의 온도 상승



[Review] 열전달의 기본 메커니즘

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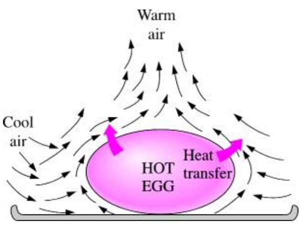
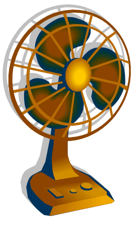
❑ 대류 (Convection)

- 고체 표면과 주변의 움직이는 액체 혹은 기체와의 열전달 형태
- 전도와 유체 유동의 복합적인 효과 포함
- 자연대류(free convection): 유체의 온도 차이로 의한 밀도 차이에 의해 발생되는 부력이 유체의 유동 유발

  예) 욕조 내의 온도 변화
- 강제대류(forced convection): 외부 요인에 의해 표면 위의 유동이 강제적으로 발생될 때 발생

  예) 선풍기 바람에 의한 체온 감소 현상





[Review] 열전달의 기본 메커니즘

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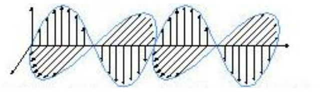
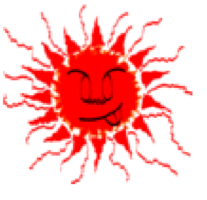
❑ 복사 (Radiation)

- 원자나 분자에서의 전자 배치의 변화로 인하여 전자기파 또는 광자의 형태로 물체로부터 방사되는 에너지에 의한 열전달
- 전도, 대류와 달리 중간 매개체가 필요하지 않음
- 열복사(Thermal radiation): 물체의 온도에 따라 물체로부터 에너지가 복사되는 현상. 온도가 높아질수록 복사량 증가.
- 예) 태양에너지가 지구에 도달하는 현상

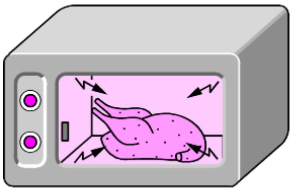




[Review] 열전달의 기본 메커니즘

❑ 열전달 사례: 전자레인지의 가열 원리

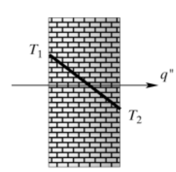
- 마그네트론(Magnetron)이라고 하는 마이크로웨이브 튜브에서 발생되는 전자기 복사 에너지를 흡수하여 음식이 요리됨.
- 마그네트론에서 방출되는 복사 에너지는 전기 에너지가 특정한 파장의 전자기 복사로 변환되는 형태 (열복사와 차이)
- 복사된 전자기파는 금속 표면에 의해 반사되고 유리, 세라믹, 플라스틱으로 된 조리 기구를 투과하여 음식물(물, 당분, 지방 등)의 분자에 흡수되어 내부에너지로 변환됨.
- 전자기파의 흡수는 주로 음식물 표면부에서 발생되어 표면부위의 온도가 일차적으로 상승
  ➔ 열전도에 의한 내부 온도 상승



Heat Transfer Problems

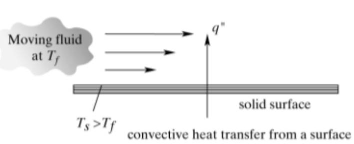
❑ Heat Transfer Modes (2D)

- Conduction: Fourier's law



conduction through a solid object

- Convection (Newton's law of cooling)

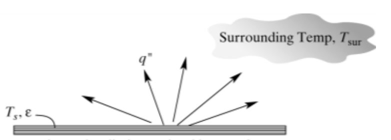


convective heat transfer from a surface

- Radiation :

  ε : emissivity (between 0 and 1)

  σ : Stefan-boltzman constant (5.67×10^{-8} W/m^2-K^4)



thermal radiation emitted by a surface

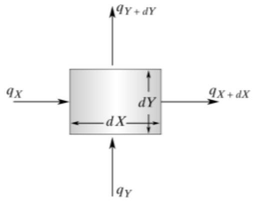
Heat Transfer Problems

❑ Energy Conservation Principle (2D)

- Energy conservation principle:

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{generation} = \dot{E}_{stored}$$

- Energy conservation in a small volume



$$q_X + q_Y - \left(q_X + \frac{\partial q_X}{\partial X} dX + q_Y + \frac{\partial q_Y}{\partial Y} dY \right) + \dot{q}\, dXdY = \rho c dXdY \frac{\partial T}{\partial t}$$

$(\dot{q} : heat\ generation\ per\ unit\ volume)$

Heat Transfer Problems

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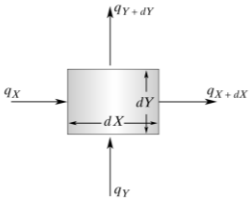
❑ Energy Conservation Principle (2D)

- Use of Fourier's law:

$$q_X = -kA \frac{\partial T}{\partial X} = -k_X dY (1) \frac{\partial T}{\partial X}$$

$$q_Y = -kA \frac{\partial T}{\partial Y} = -k_Y dX (1) \frac{\partial T}{\partial Y}$$



⇨

- Steady state:

$$-\frac{\partial}{\partial X}\left(-k_X dY \frac{\partial T}{\partial X} \right) dX - \frac{\partial}{\partial Y}\left(-k_Y dX \frac{\partial T}{\partial Y} \right) dY + \dot{q}\, dXdY = \rho c dXdY \frac{\partial T}{\partial t}$$ ⇨

Heat Transfer Problems

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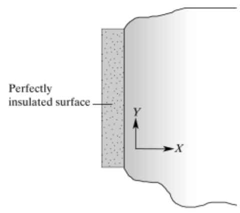
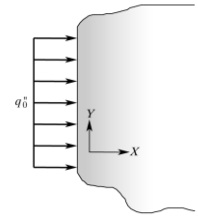
❑ Boundary Conditions

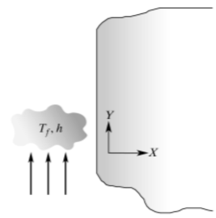
- Adiabatic condition (perfect insulation):
- Constant heat flux condition:
- Convection condition:
- Constant temperature condition:



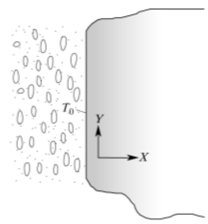
Surface insulation



Constant heat flux



Convection



Constant temperature

Heat Transfer Problems – FE Formulation

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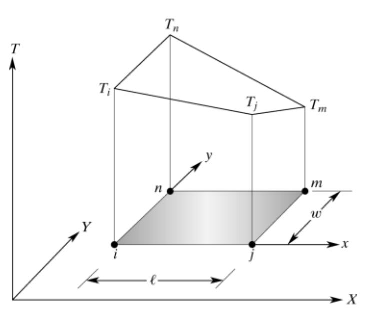
❑ Formulation with Rectangular (Quadrilateral) Elements

- Temperature distribution for an element

$$T^{(e)} = \begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

- Shape functions



❑ Formulation with Rectangular (Quadrilateral) Elements

- Galerkin approach for the heat diffusion equation

$$R_i^{(e)} = \int_A S_i \left(k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + \dot{q} \right) dA$$

$$R_j^{(e)} = \int_A S_j \left(k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + \dot{q} \right) dA$$

$$R_m^{(e)} = \int_A S_m \left(k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + \dot{q} \right) dA$$

$$R_n^{(e)} = \int_A S_n \left(k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + \dot{q} \right) dA$$

$$[\mathbf{S}]^T = \begin{Bmatrix} S_i \\ S_j \\ S_k \\ S_n \end{Bmatrix}$$

Heat Transfer Problems – FE Formulation
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❑ Formulation with Rectangular (Quadrilateral) Elements

- Let $C_1 = k_x,\ C_2 = k_y,\ C_3 = \dot{q}$

- By applying the chain rule

$$\frac{\partial}{\partial x}\left([\mathbf{S}]^T \frac{\partial T}{\partial x} \right) = [\mathbf{S}]^T \frac{\partial^2 T}{\partial x^2} + \frac{\partial [\mathbf{S}]^T}{\partial x} \frac{\partial T}{\partial x}$$

$$\int_A [\mathbf{S}]^T \left(C_1 \frac{\partial^2 T}{\partial x^2} \right) dA = \int_A C_1 \frac{\partial}{\partial x}\left([\mathbf{S}]^T \frac{\partial T}{\partial x} \right) dA - \int_A C_1 \left(\frac{\partial [\mathbf{S}]^T}{\partial x} \frac{\partial T}{\partial x} \right) dA$$

$$\int_A [\mathbf{S}]^T \left(C_2 \frac{\partial^2 T}{\partial x^2} \right) dA = \int_A C_2 \frac{\partial}{\partial x}\left([\mathbf{S}]^T \frac{\partial T}{\partial x} \right) dA - \int_A C_2 \left(\frac{\partial [\mathbf{S}]^T}{\partial x} \frac{\partial T}{\partial x} \right) dA$$

❑ Formulation with Rectangular (Quadrilateral) Elements

- Calculation of each term

$$\int_A [\mathbf{S}]^T \left(C_1 \frac{\partial^2 T}{\partial x^2} \right) dA = \int_A C_1 \frac{\partial}{\partial x} \left([\mathbf{S}]^T \frac{\partial T}{\partial x} \right) dA - \quad$$

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x} \begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix} = \frac{1}{\ell w} \begin{bmatrix} (-w+y) & (w-y) & y & -y \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

$$\frac{\partial [\mathbf{S}]^T}{\partial x} = \frac{\partial}{\partial x} \begin{Bmatrix} S_i \\ S_j \\ S_m \\ S_n \end{Bmatrix} = \frac{1}{\ell w} \begin{Bmatrix} -w+y \\ w-y \\ y \\ -y \end{Bmatrix}$$

Heat Transfer Problems – FE Formulation

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❑ Formulation with Rectangular (Quadrilateral) Elements

- Calculation of each term (cont'd)

$$= \frac{-C_1 w}{6\ell} \begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

$$\int_0^\ell \int_0^w \frac{1}{w\ell} (-w+y)^2 dy dx = \frac{1}{(w\ell)^2} \int_0^\ell \left(w^2 w + \frac{w^3}{3} - 2w \frac{w^2}{2} \right) dx = \frac{1}{(w\ell)^2} \int_0^\ell \frac{w^3}{3} dx = \frac{1}{(w\ell)^2} \frac{w^3 \ell}{3} = \frac{1}{3} \frac{w}{\ell}$$

Heat Transfer Problems – FE Formulation

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❑ Formulation with Rectangular (Quadrilateral) Elements

- Calculation of each term (cont'd)

$$\int_A [\mathbf{S}]^T \left(C_1 \frac{\partial^2 T}{\partial x^2} \right) dA + \boxed{} + \boxed{} = 0$$

$$-\int_A C_2 \left(\frac{\partial [\mathbf{S}]^T}{\partial y} \frac{\partial T}{\partial y} \right) dA = -\frac{C_1 w}{6\ell} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

$$\int_A [\mathbf{S}]^T C_3 dA = C_3 \int_A \begin{Bmatrix} S_i \\ S_j \\ S_m \\ S_n \end{Bmatrix} dA = \frac{C_3 A}{4} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Heat Transfer Problems – FE Formulation

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❑ Formulation with Rectangular (Quadrilateral) Elements

- [Review] Galerkin equation

$$\int_A [\mathbf{S}]^T \left(C_1 \frac{\partial^2 T}{\partial x^2} \right) dA + \int_A [\mathbf{S}]^T \left(C_2 \frac{\partial^2 T}{\partial x^2} \right) dA + \boxed{\int_A [\mathbf{S}]^T C_3 dA} = 0$$

$$\int_A [\mathbf{S}]^T \left(C_1 \frac{\partial^2 T}{\partial x^2} \right) dA = \underline{\int_A C_1 \frac{\partial}{\partial x} \left([\mathbf{S}]^T \frac{\partial T}{\partial x} \right) dA} - \boxed{\int_A C_1 \left(\frac{\partial [\mathbf{S}]^T}{\partial x} \frac{\partial T}{\partial x} \right) dA}$$

$$\int_A [\mathbf{S}]^T \left(C_2 \frac{\partial^2 T}{\partial x^2} \right) dA = \underline{\int_A C_2 \frac{\partial}{\partial x} \left([\mathbf{S}]^T \frac{\partial T}{\partial x} \right) dA} - \boxed{\int_A C_2 \left(\frac{\partial [\mathbf{S}]^T}{\partial x} \frac{\partial T}{\partial x} \right) dA}$$

- By applying Green's theorem

τ: The element boundary

θ: The angle to the unit normal

❑ Formulation with Rectangular (Quadrilateral) Elements

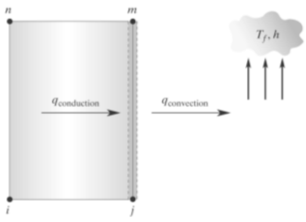
- Energy conservation in x-direction

$$q_{\text{cond.}} = q_{\text{conv.}} \quad \Rightarrow \quad -k\frac{\partial T}{\partial x} = h\left(T - T_f\right)$$

- Green's theorem in x-direction

$$\int_A C_1 \frac{\partial}{\partial x}\left([\mathbf{S}]^T \frac{\partial T}{\partial x}\right) dA = \int_\tau C_1 [\mathbf{S}]^T \frac{\partial T}{\partial x}\cos\theta\, d\tau$$

$$\Rightarrow \quad \int_\tau C_1 [\mathbf{S}]^T \frac{\partial T}{\partial x}\cos\theta\, d\tau = \int_\tau k[\mathbf{S}]^T \frac{\partial T}{\partial x}\cos\theta\, d\tau = -\int_\tau h[\mathbf{S}]^T\left(T - T_f\right)\cos\theta\, d\tau$$



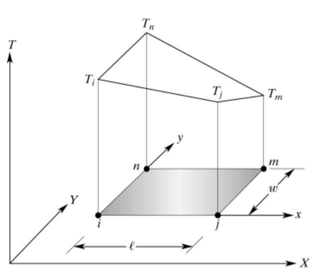
Convective boundary conditions

❑ Formulation with Rectangular (Quadrilateral) Elements

- Convective boundary conditions along different edges of the rectangular element: related to the stiffness matrix

$$[\mathbf{K}]^{(e)} = \frac{h\ell_{jm}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

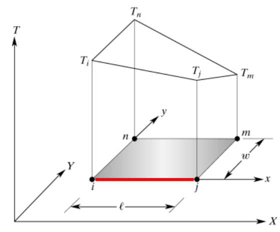
$$[\mathbf{K}]^{(e)} = \frac{h\ell_{mn}}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad [\mathbf{K}]^{(e)} = \frac{h\ell_{ni}}{6}\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$



$$\ell_{ij} = \ell_{mn} = \ell, \quad \ell_{jm} = \ell_{in} = w$$

❑ Formulation with Rectangular (Quadrilateral) Elements

- How to get the matrix?



$$\int_\tau h[\mathbf{S}]^T T\cos\theta\, d\tau \;\Rightarrow\; [\mathbf{K}]^{(e)} = \frac{h\ell_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\int_\tau h[\mathbf{S}]^T T d\tau = \int_\tau h \begin{Bmatrix} S_i \\ S_j \\ S_m \\ S_n \end{Bmatrix} \overbrace{\begin{bmatrix} S_i & S_j & S_m & S_n \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}}^{T} d\tau = h\int_\tau \begin{bmatrix} S_i^2 & S_jS_i & S_mS_i & S_nS_i \\ S_iS_j & S_j^2 & S_mS_j & S_nS_j \\ S_iS_m & S_jS_m & S_m^2 & S_nS_m \\ S_iS_n & S_jS_n & S_mS_n & S_n^2 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix} d\tau$$

❑ Formulation with Rectangular (Quadrilateral) Elements

- Along the edge *ij*: $\eta = -1$, $-1 \le \xi \le 1$

$$\int_\tau h[\mathbf{S}]^T T d\tau = \frac{h\ell_{ij}}{2}\int_\tau \begin{bmatrix} \left(\frac{1}{4}(1-\xi)(1-\eta)\right)^2 & \left(\frac{1}{16}(1-\xi)(1-\eta)(1+\xi)(1-\eta)\right) & 0 & 0 \\ \left(\frac{1}{16}(1-\xi)(1-\eta)(1+\xi)(1-\eta)\right) & \left(\frac{1}{4}(1+\xi)(1-\eta)\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix} d\xi$$

$$\Rightarrow \int_\tau h[\mathbf{S}]^T T d\tau = \frac{h\ell_{ij}}{2}\int_{-1}^{1} \begin{bmatrix} \left(\frac{1}{2}(1-\xi)\right)^2 & \left(\frac{1}{4}(1-\xi)(1+\xi)\right) & 0 & 0 \\ \left(\frac{1}{4}(1-\xi)(1+\xi)\right) & \left(\frac{1}{4}(1+\xi)\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} d\xi \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix} = \frac{h\ell_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_m \\ T_n \end{Bmatrix}$$

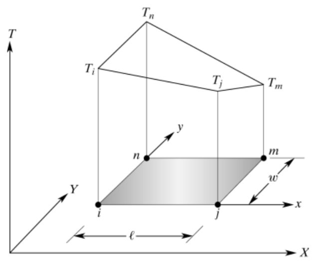
❑ Formulation with Rectangular (Quadrilateral) Elements

- Elemental thermal load matrix

$$= -\int_{\tau} h[\mathbf{S}]^T T \cos\theta \, d\tau +$$

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f \ell_{ij}}{2}\begin{Bmatrix}1\\1\\0\\0\end{Bmatrix} \qquad \{\mathbf{F}\}^{(e)} = \frac{hT_f \ell_{jm}}{2}\begin{Bmatrix}0\\1\\1\\0\end{Bmatrix}$$

$$\{\mathbf{F}\}^{(e)} = \frac{hT_f \ell_{mn}}{2}\begin{Bmatrix}0\\0\\1\\1\end{Bmatrix} \qquad \{\mathbf{F}\}^{(e)} = \frac{hT_f \ell_{ni}}{2}\begin{Bmatrix}1\\0\\0\\1\end{Bmatrix}$$



$$\ell_{ij} = \ell_{mn} = \ell, \quad \ell_{jm} = \ell_{in} = w$$

❑ Formulation with Rectangular (Quadrilateral) Elements

- The conductance matrix for a bilinear rectangular element

$$\Rightarrow \quad [\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{F}\}$$

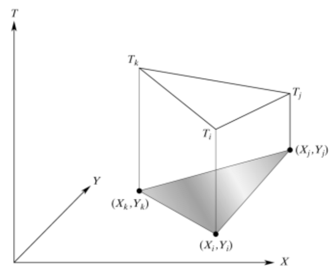
- Thermal load vector due to the heat generation :

- Thermal load vector due to the convection (see the previous page)

❑ Formulation with Triangular Elements

- Shape function

$$T^{(e)} = \begin{bmatrix} S_i & S_j & S_k \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

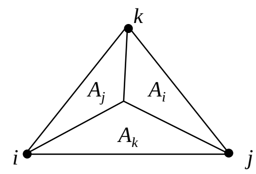


- Area-based coordinate

$$\alpha_i = X_j Y_k - X_k Y_j \qquad \beta_i = Y_j - Y_k \qquad \delta_i = X_k - X_j$$

$$\alpha_j = X_k Y_i - X_i Y_k \qquad \beta_j = Y_k - Y_i \qquad \delta_j = X_i - X_k$$

$$\alpha_k = X_i Y_j - X_j Y_i \qquad \beta_k = Y_i - Y_j \qquad \delta_k = X_j - X_i$$



Heat Transfer Problems

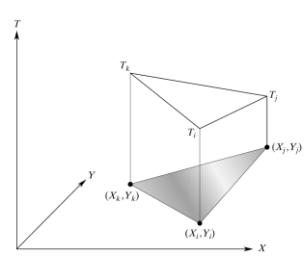
서울과학기술대학교 SEOUL NATIONAL UNIVERSITY OF SCIENCE & TECHNOLOGY

❑ Formulation with Triangular Elements

- Residual equations for a triangular element (Galerkin approach)

- Applying the chain rule (2nd derivative => 1st derivative)

$$-\int_A C_1 \left(\frac{\partial [\mathbf{S}]^T}{\partial X} \frac{\partial T}{\partial X} \right) dA$$



❑ Formulation with Triangular Elements

- Substituting for the derivatives

$$-\int_A C_1\left(\frac{\partial[\mathbf{S}]^T}{\partial X}\frac{\partial T}{\partial X}\right)dA = -C_1\int_A \frac{1}{4A^2}\begin{Bmatrix}\beta_i\\ \beta_j\\ \beta_k\end{Bmatrix}\begin{bmatrix}\beta_i & \beta_j & \beta_k\end{bmatrix}\begin{Bmatrix}T_i\\ T_j\\ T_k\end{Bmatrix}dA$$

- Same calculation for the 2nd derivative with respect to Y (Term 2)

$$-\int_A C_2\left(\frac{\partial[\mathbf{S}]^T}{\partial Y}\frac{\partial T}{\partial Y}\right)dA = -\frac{C_2}{4A^2}\begin{bmatrix}\delta_i^2 & \delta_i\delta_j & \delta_i\delta_k\\ \delta_i\delta_j & \delta_j^2 & \delta_j\delta_k\\ \delta_i\delta_k & \delta_j\delta_k & \delta_k^2\end{bmatrix}\begin{Bmatrix}T_i\\ T_j\\ T_k\end{Bmatrix}$$

❑ Formulation with Triangular Elements

- Thermal load matrix due to the heat generation (Term 3)

- Convective boundary conditions along the edges of triangular element

$$-\int_\tau h[\mathbf{S}]^T(T-T_f)\cos\theta\,d\tau = \qquad + \int_A h[\mathbf{S}]^T T_f\cos\theta\,d\tau$$

$$-\int_\tau h[\mathbf{S}]^T(T-T_f)\sin\theta\,d\tau = \qquad + \int_A h[\mathbf{S}]^T T_f\sin\theta\,d\tau$$

$\ell_{ij},\ \ell_{jk},\ \ell_{ki}$: the length of the three sides of the triangular element

❑ Formulation with Triangular Elements

- Elemental thermal load vector

$$-\int_{\tau} h[\mathbf{S}]^T (T - T_f)\cos\theta \, d\tau = -\int_A h[\mathbf{S}]^T T \cos\theta \, d\tau +$$

$$-\int_{\tau} h[\mathbf{S}]^T (T - T_f)\sin\theta \, d\tau = -\int_A h[\mathbf{S}]^T T \sin\theta \, d\tau +$$

- Elemental conductance matrix

Cf) Ex. 7.1 (p. 279~286)

X-dir. conduction Y-dir. conduction