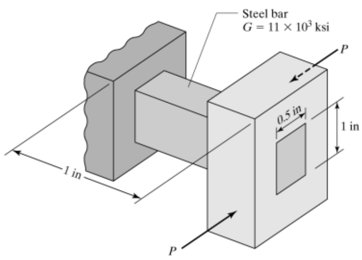
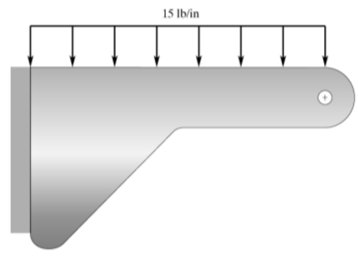
Chap. 7 Analysis of 2d Solid Mechanics Problems





Torsion Member (Arbitrary Cross-Section)

❑ [Review] Torsion of a Circular Shaft

- The shear stress distribution

$$\tau = \frac{Tr}{J}$$

T: the applied torque
r: the radial distance
J: the polar moment of inertia

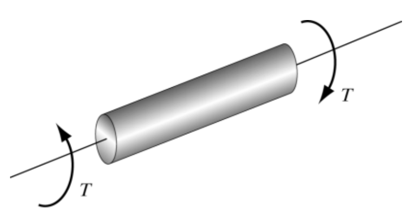
- The angle of twist

$$\theta = \frac{TL}{JG}$$

L: the length of the member
G: the shear modulus

- The maximum shear stress (Radius: R)

$$\tau_{\max} = \frac{TR}{\frac{1}{2}\pi R^4} = \frac{2T}{\pi R^3}$$



Torsion Member (Arbitrary Cross-Section)

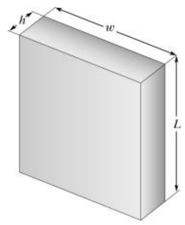
❑ Torsion of a Bar with a Rectangular Cross Section

- The angle of twist

w/h	c_1	c_2
1.0	0.208	0.141
1.2	0.219	0.166
1.5	0.231	0.196
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

- The maximum shear stress (Radius: R)

- For high aspect ratio (w/h > 10)



Torsion Member (Arbitrary Cross-Section)

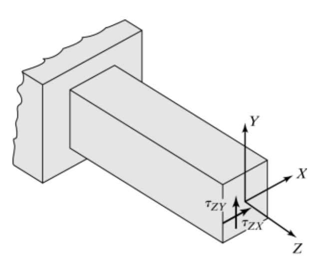
❑ Prandtl Formulation for Torsion Formulations

- The governing differential equation

where θ : the angle of twist per unit length
φ : the stress function

- The shear stress components

- The applied torque

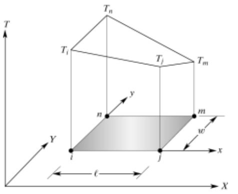


❑ [Review] 2D Heat Transfer Formulation (Rectangular elements)

- Galerkin formulation as a generalized form

$$\int_A [\mathbf{S}]^T \left(C_1 \frac{\partial^2 T}{\partial x^2} \right) dA + \int_A [\mathbf{S}]^T \left(C_2 \frac{\partial^2 T}{\partial x^2} \right) dA + \int_A [\mathbf{S}]^T C_3 dA = 0$$

$$C_1 = k_x,\ C_2 = k_y,\ C_3 = q^{\square}$$



- Elementary conductance matrix & Thermal load vector

$$[\mathbf{K}]^{(e)} = \frac{\overset{C_1}{k} w}{6l} \begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{\overset{C_2}{k} l}{6w} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix} \qquad \{\mathbf{F}\}^{(e)} = \frac{\overset{C_3}{\dot{q}} A}{4} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

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❑ Torsion vs. 2d Heat Transfer Problems

- The governing differential equation

Torsion eq. $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2G\theta = 0$$

Heat diffusion eq. $$k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} + \dot{q} = 0$$

- Elementary stiffness matrix & load vector

❑ Beam

- A structural member whose cross-sectional dimensions are relatively smaller than length
- Commonly subjected to transverse loading: Bending
- Engineering applications: buildings, bridges, automobiles, and airplane structures

❑ Truss vs. Beam

- Truss: Loads are applied only at joints
- Beam: Loads are applied at any locations

❑ Approximations

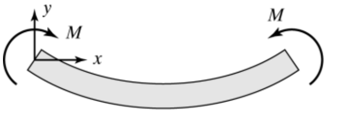
- Normal planes to the neutral axis are maintained to the normal planes after deflection
- The effects of the shear stresses are neglected

❑ Beam Formulation (Euler Beam Theory)

- The flexure formula

$$\sigma = -\frac{My}{I}$$

M: the applied bending moment
y: distance from the neutral axis
I: the second moment of inertia



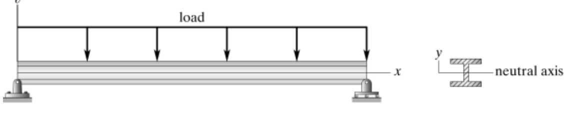
Positive bending and
Positive curvature

- The deflection of the neutral axis (v)

$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$

$$EI\frac{d^3\upsilon}{dx^3} = \frac{dM(x)}{dx} = V(x)$$

$$EI\frac{d^4\upsilon}{dx^4} = \frac{dV(x)}{dx} = w(x)$$



❑ FE Formulation – Elementary Stiffness Matrix

- The strain energy for an arbitrary beam element (e)

- Recognizing the integral $\int_A y^2 dA$ is the second moment of inertia (I)

- Approximation of the deflection (v) with the 3rd order polynomial

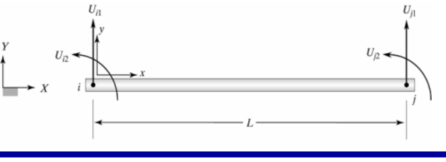
$$\upsilon = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

❑ FE Formulation – Elementary Stiffness Matrix

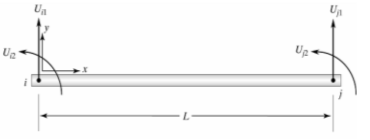
- The element's end conditions (nodal values) $\upsilon = c_1 + c_2 x + c_3 x^2 + c_4 x^3$

Node	Node *i* (*x* = 0)	Node *j* (*x* = L)
Displacement	$\upsilon = c_1 = U_{i1}$	$\upsilon = c_1 + c_2 L + c_3 L^2 + c_4 L^3 = U_{j1}$
Slope	$\left.\frac{d\upsilon}{dx}\right\vert_{x=0} = c_2 = U_{i2}$	$\left.\frac{d\upsilon}{dx}\right\vert_{x=L} = c_2 + 2c_3 L + 3c_4 L^2 = U_{j2}$

- 4 equations with four unknowns: solve for c_1, c_2, c_3, c_4



❑ FE Formulation – Elementary Stiffness Matrix



- Expression using the shape function

$$\upsilon = S_{i1}U_{i1} + S_{i2}U_{i2} + S_{j1}U_{j1} + S_{j2}U_{j2}$$

- End conditions: Verification
  - $x=0$ 인 절점 i 에서의 형상함수 $\quad S_{i1}=1,\ S_{i2}=S_{j1}=S_{j2}=0$
  - $x=0$ 인 절점 i 에서의 형상함수의 기울기 $\quad \dfrac{dS_{i2}}{dx}=1,\ \dfrac{dS_{i1}}{dx}=\dfrac{dS_{j1}}{dx}=\dfrac{dS_{j2}}{dx}=0$
  - $x=L$ 인 절점 j 에서의 형상함수 $\quad S_{j1}=1,\ S_{i1}=S_{i2}=S_{j2}=0$
  - $x=L$ 인 절점 j 에서의 형상함수의 기울기 $\quad \dfrac{dS_{j2}}{dx}=1,\ \dfrac{dS_{i1}}{dx}=\dfrac{dS_{i2}}{dx}=\dfrac{dS_{j1}}{dx}=0$

❑ FE Formulation – Elementary Stiffness Matrix

- Governing equation in terms of the shape functions

$$EI\frac{d^2\upsilon}{dx^2} = M(x)$$

- The 2nd derivatives of the shape functions

$$D_{i1} = \frac{d^2S_{i1}}{dx^2} = -\frac{6}{L^2} + \frac{12x}{L^3} \qquad D_{i2} = \frac{d^2S_{i2}}{dx^2} = -\frac{4}{L} + \frac{6x}{L^2}$$

$$D_{j1} = \frac{d^2S_{j1}}{dx^2} = \frac{6}{L^2} - \frac{12x}{L^3} \qquad D_{j2} = \frac{d^2S_{j2}}{dx^2} = -\frac{2}{L} + \frac{6x}{L^2}$$

❑ FE Formulation – Elementary Stiffness Matrix

- The strain energy for an arbitrary beam element

$$\Lambda^{(e)} = \frac{EI}{2}\int_0^L \left(\frac{\partial^2 \upsilon}{\partial x^2}\right)^2 dx$$

since $\left(\frac{\partial^2 \upsilon}{\partial x^2}\right)^2 = ([\mathbf{D}]\{\mathbf{U}\})([\mathbf{D}]\{\mathbf{U}\}) = \{\mathbf{U}\}^T[\mathbf{D}]^T[\mathbf{D}]\{\mathbf{U}\}$

- Minimum total potential energy principle

$$\Pi = \sum \Lambda^{(e)} - \sum FU \quad \Rightarrow \quad \frac{\partial \Pi}{\partial U_k} = \frac{\partial}{\partial U_k}\sum \Lambda^{(e)} - \frac{\partial}{\partial U_k}\sum FU = 0 \qquad k=1,2,3,4$$

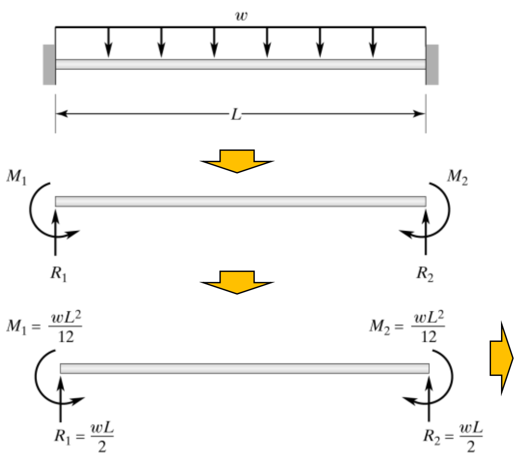
❑ FE Formulation – Elementary Stiffness Matrix

- The strain energy term

- The elementary stiffness matrix

$$[\mathbf{K}]^{(e)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

❑ FE Formulation – Elementary Load Vector



Calculation of the equivalent nodal loads

→ Reverse the signs

(pp. 326 ~ 327)

Beams and Frames

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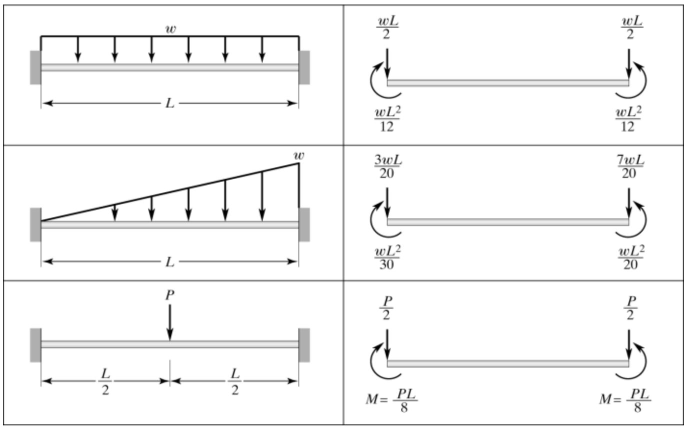
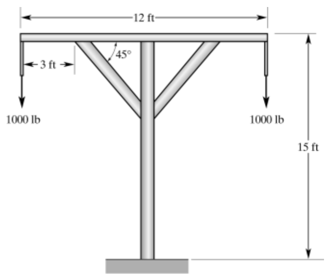
❑ FE Formulation – Elementary Load Vector



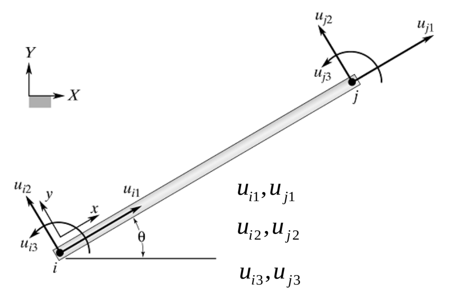
❑ Frames

- Structural members rigidly connected with welded or bolted joints
- Displacements (DOFs): [rotation and lateral displacement] + axial deformation

DOFs of a beam



Problem 4-11



❑ FE Formulation for a Frame

- Coordinate transformation – use of the transformation matrix

$$[\mathbf{u}] = [\mathbf{T}][\mathbf{U}]$$

Local DOF ↑ ↑ Global DOF

❑ FE Formulation for a Frame

- Elementary stiffness matrix

$$
[\mathbf{K}]_{xy}^{(e)} = \frac{EI}{L^3}
\begin{array}{c}
\begin{matrix} u_{i1} & u_{i2} & u_{i3} & u_{j1} & u_{j2} & u_{j3} \end{matrix} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 6L & 0 & -12 & 6L \\
0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -12 & -6L & 0 & 12 & -6L \\
0 & 6L & 2L^2 & 0 & -6L & 4L^2
\end{bmatrix}
\end{array}
\begin{matrix} u_{i1} \\ u_{i2} \\ u_{i3} \\ u_{j1} \\ u_{j2} \\ u_{j3} \end{matrix}
$$

$$
[\mathbf{K}]_{axial}^{(e)} =
\begin{array}{c}
\begin{matrix} u_{i1} & u_{i2} & u_{i3} & u_{j1} & u_{j2} & u_{j3} \end{matrix} \\
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{matrix} u_{i1} \\ u_{i2} \\ u_{i3} \\ u_{j1} \\ u_{j2} \\ u_{j3} \end{matrix}
$$

❑ FE Formulation for a Frame

- Elementary stiffness matrix in local coordinate system

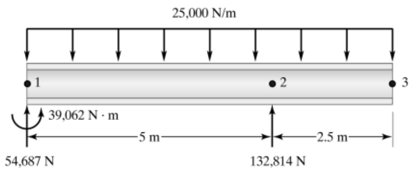
$$
[\mathbf{K}]_{xy}^{(e)} =
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
$$

- Elementary stiffness matrix in local coordinate system

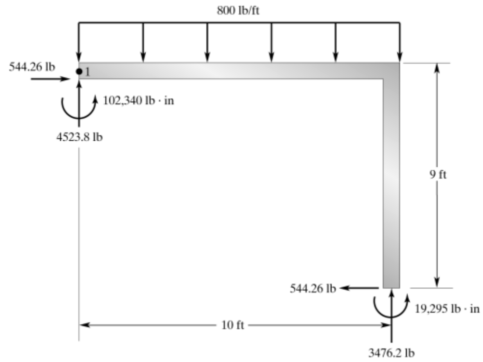
Beams and Frames

❑ Examples

Ex 8.1 Beam (pp. 328 ~ 331)



Ex 8.2 Frame (pp. 334 ~ 338)



Plane Stress Formulation

❑ [Review] 2D Structural Problems

- Plans Stress

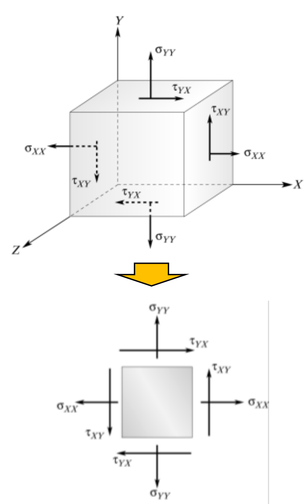
$$\sigma_{zz} = \tau_{zx} = \tau_{zx} = 0 \quad \Rightarrow \quad [\sigma]^T = [\sigma_{xx} \;\; \sigma_{yy} \;\; \tau_{xy}]^T$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

- Plane Strain

$$\varepsilon_{zz} = \gamma_{zx} = \gamma_{zx} = 0 \quad \Rightarrow \quad [\varepsilon]^T = [\varepsilon_{xx} \;\; \varepsilon_{yy} \;\; \gamma_{xy}]^T$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$



❑ [Review] Plane Stress Problems

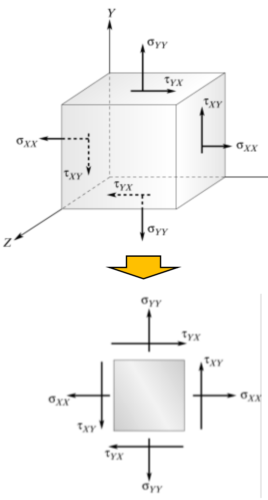
$$[\sigma]^T = [\sigma_{XX} \quad \sigma_{YY} \quad \sigma_{ZZ} \quad \tau_{XY} \quad \tau_{YZ} \quad \tau_{XZ}]$$

$$\sigma_{zz} = \tau_{zx} = \tau_{zx} = 0 \quad \Rightarrow \quad [\sigma]^T = [\sigma_{xx} \ \sigma_{yy} \ \tau_{xy}]^T$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\Rightarrow \quad \{\boldsymbol{\sigma}\} = [\mathbf{v}]\{\boldsymbol{\varepsilon}\}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



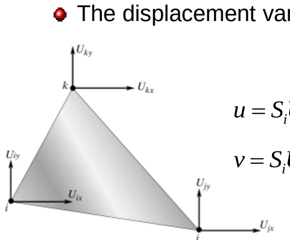
❑ FE Formulation – Elementary Stiffness Matrix

- The strain energy for an arbitrary element under biaxial loading

$$\Lambda^{(e)} = \frac{1}{2}\int_V \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy}\right) dV \qquad \Lambda^{(e)} = \frac{1}{2}\int_V [\boldsymbol{\sigma}]^T \{\boldsymbol{\varepsilon}\}\, dV$$

⇨ (Hooke's law)

- The displacement variable in terms of linear triangular shape functions



$$u = S_i U_{ix} + S_j U_{jx} + S_k U_{kx}$$
$$v = S_i U_{iy} + S_j U_{jy} + S_k U_{ky}$$

$$\Rightarrow \quad \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} S_i & 0 & S_j & 0 & S_k & 0 \\ 0 & S_i & 0 & S_j & 0 & S_k \end{bmatrix} \begin{Bmatrix} U_{ix} \\ U_{iy} \\ U_{jx} \\ U_{jy} \\ U_{kx} \\ U_{ky} \end{Bmatrix}$$

❑ FE Formulation – Elementary Stiffness Matrix

- The strain-displacement relation

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(S_i U_{ix} + S_j U_{jx} + S_k U_{kx}\right) = \frac{1}{2A}\left[\beta_i U_{ix} + \beta_j U_{jx} + \beta_k U_{kx}\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(S_i U_{iy} + S_j U_{jy} + S_k U_{ky}\right) = \frac{1}{2A}\left[\delta_i U_{iy} + \delta_j U_{jy} + \delta_k U_{ky}\right]$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}\left[\delta_i U_{ix} + \beta_i U_{iy} + \delta_j U_{jx} + \beta_j U_{jy} + \delta_k U_{kx} + \beta_k U_{ky}\right]$$

$$\Rightarrow \ \{\boldsymbol{\varepsilon}\} = [\mathbf{B}]\{\mathbf{U}\}$$

Plane Stress Formulation

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❑ FE Formulation – Elementary Stiffness Matrix

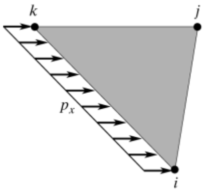
- The strain energy equation

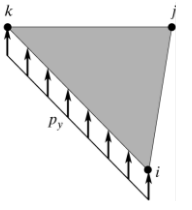
- Differentiating wrt the nodal displacement

$$\frac{\partial \Lambda^{(e)}}{\partial U_k} = \frac{\partial}{\partial U_k}\left(\frac{1}{2}\int_V [\mathbf{U}]^T [\mathbf{B}]^T [\mathbf{v}][\mathbf{B}][\mathbf{U}]\,dV\right) \qquad k = 1,\ 2, \ldots, 6$$

- The elementary stiffness matrix

❑ FE Formulation – Elementary Load Vector





- The work done by concentrated loads

$$W^{(e)} = \{\mathbf{U}\}^T \{\mathbf{Q}\}$$

- The work done by a distributed load with p_x and p_y components

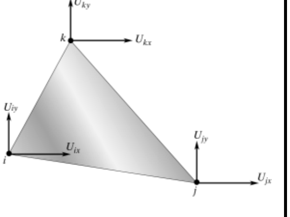
$$W^{(e)} = \int_A \left(u p_x + v p_y \right) dA$$

- The work done by the distributed load in a triangular element

❑ FE Formulation – Elementary Load Vector

- A concentrated load vector for a triangular element

$$W^{(e)} = \{\mathbf{U}\}^T \{\mathbf{Q}\} \xrightarrow{\text{differentiating}} \{\mathbf{F}\}^{(e)} = \begin{Bmatrix} Q_{ix} \\ Q_{iy} \\ Q_{jx} \\ Q_{jy} \\ Q_{kx} \\ Q_{ky} \end{Bmatrix}$$



- The differentiation of the work done by the distributed load

where

Plane Stress Formulation

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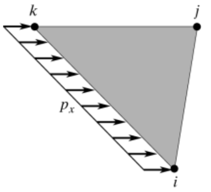
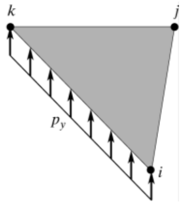
❑ FE Formulation – Elementary Load Vector

- Integrating along the *ki*-edge ($S_j = 0$) $\{\mathbf{F}\}^{(e)} = \int_A [\mathbf{S}]^T \{\mathbf{p}\} dA$





along the *ij*-edge $[\mathbf{F}]^{(e)} = \dfrac{tL_{ij}}{2}\begin{Bmatrix} p_x \\ p_y \\ p_x \\ p_y \\ 0 \\ 0 \end{Bmatrix}$

along the *jk*-edge $[\mathbf{F}]^{(e)} = \dfrac{tL_{jk}}{2}\begin{Bmatrix} 0 \\ 0 \\ p_x \\ p_y \\ p_x \\ p_y \end{Bmatrix}$

Plane Stress Formulation

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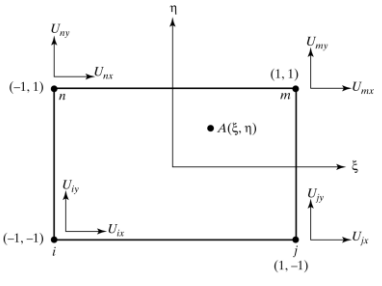
❑ Isoparametric Formulation – Q4 Element

- Displacement field within an element

$$u = S_i U_{ix} + S_j U_{jx} + S_m U_{mx} + S_n U_{nx}$$
$$v = S_i U_{iy} + S_j U_{jy} + S_m U_{my} + S_n U_{ny}$$



Position within the element

$$x = S_i x_i + S_j x_j + S_m x_m + S_n x_n$$
$$y = S_i y_i + S_j y_j + S_m y_m + S_n y_n$$

❑ Isoparametric Formulation – Q4 Element

- Using the Jacobian of the coord. transformation

$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \xi}\left[S_i x_i + S_j x_j + S_m x_m + S_n x_n\right] & \frac{\partial}{\partial \xi}\left[S_i y_i + S_j y_j + S_m y_m + S_n y_n\right] \\ \frac{\partial}{\partial \eta}\left[S_i x_i + S_j x_j + S_m x_m + S_n x_n\right] & \frac{\partial}{\partial \eta}\left[S_i y_i + S_j y_j + S_m y_m + S_n y_n\right] \end{bmatrix}$$

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❑ Isoparametric Formulation – Q4 Element

- Jacobian matrix for a Q4 element

$$[\mathbf{J}] = \frac{1}{4}\begin{bmatrix} \left[-(1-\eta)x_i + (1-\eta)x_j + (1+\eta)x_m - (1+\eta)x_n\right] \\ \left[-(1-\xi)x_i - (1+\xi)x_j + (1+\xi)x_m + (1-\xi)x_n\right] \end{bmatrix}$$

$$\begin{bmatrix} \left[-(1-\eta)y_i + (1-\eta)y_j + (1+\eta)y_m - (1+\eta)y_n\right] \\ \left[-(1-\xi)y_i - (1+\xi)y_j + (1+\xi)y_m + (1-\xi)y_n\right] \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$[\mathbf{J}]^{-1} = \frac{1}{J_{11}J_{22} - J_{12}J_{21}}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

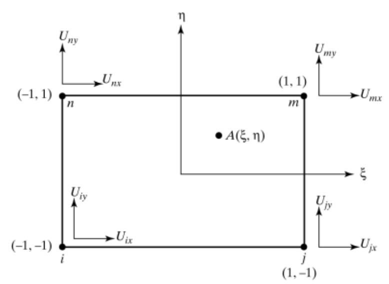
❑ Isoparametric Formulation – Q4 Element

- The strain energy of an element

$$\Lambda^{(e)} = \frac{1}{2}\int_V \{\boldsymbol{\varepsilon}\}^T [\mathbf{v}]\{\boldsymbol{\varepsilon}\}\, dV = \frac{1}{2}(t_e)\int_A \{\boldsymbol{\varepsilon}\}^T [\mathbf{v}]\{\boldsymbol{\varepsilon}\}\, dA$$

(t_e: element thickness)

- The strain-displacement relation



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❑ Isoparametric Formulation – Q4 Element

- The strain-displacement relation (cont'd)

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \qquad \begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

❑ Isoparametric Formulation – Q4 Element

- The strain-displacement relation (cont'd)

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \frac{1}{4}\overbrace{\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & -(1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix}}^{[\mathbf{D}]} \overbrace{\begin{Bmatrix} U_{ix} \\ U_{iy} \\ U_{jx} \\ U_{jy} \\ U_{mx} \\ U_{my} \\ U_{nx} \\ U_{ny} \end{Bmatrix}}^{\{\mathbf{U}\}}$$

❑ Isoparametric Formulation – Q4 Element

- The strain energy of an element (in natural coordinate)

$$\Lambda^{(e)} = \frac{1}{2}(t_e)\int_A \{\boldsymbol{\varepsilon}\}^T [\mathbf{v}]\{\boldsymbol{\varepsilon}\}\, dA = \frac{1}{2}(t_e)\int_{-1}^{1}\int_{-1}^{1} \{\boldsymbol{\varepsilon}\}^T [\mathbf{v}]\{\boldsymbol{\varepsilon}\} \overbrace{\det \mathbf{J}\, d\xi\, d\eta}^{dA}$$

- The strain-displacement relation (matrix form)

- The elementary stiffness matrix