Sixth GE SI Edition

CHAPTER

3

MECHANICS OF MATERIALS

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Torsion

Contents

Torsional Loads on Circular Shafts (비틀림 하중)

- Torsion: a load as a form of twisting moment (eg. Transmission shafts to transmit power from the automobile engine to the rear wheels)
- Example: A steam turbine A and an electric generator B connected by a transmission shaft AB
- Turbine exerts torque *T* on the shaft ▶ Shaft transmits the torque to the generator ▶ Generator creates an equal and opposite torque *T*'

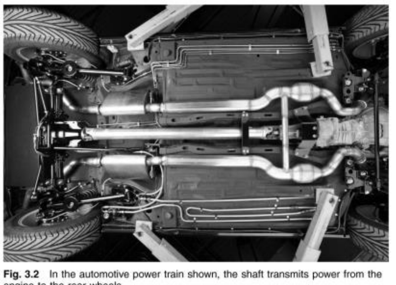
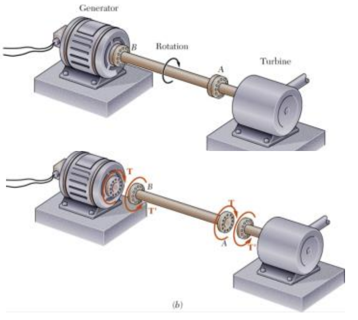
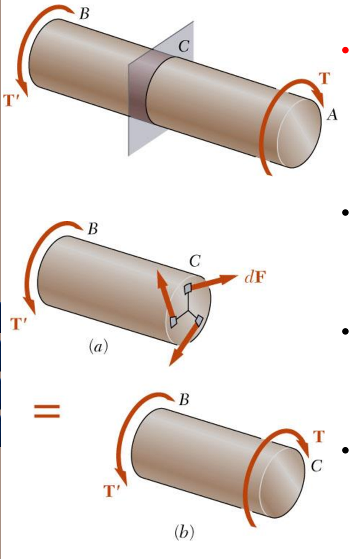


Fig. 3.2 In the automotive power train shown, the shaft transmits power from the engine to the rear wheels.

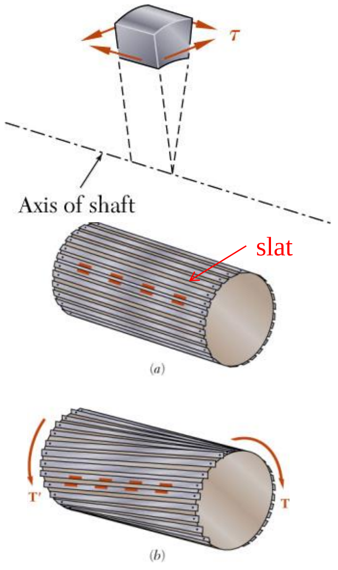


Net Torque Due to Internal Stresses



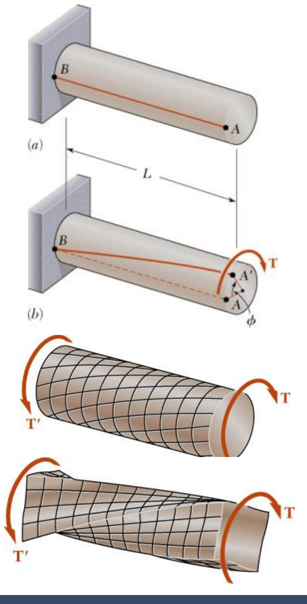
- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,
- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not.
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations.
- Unlike the normal stress due to axial loads,

Axial Shear Components



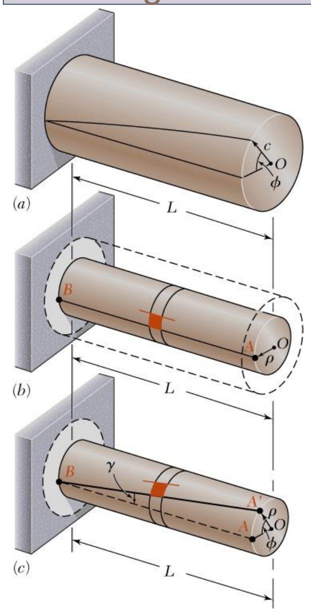
- Torque applied to shaft produces
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft.
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.
- The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Shaft Deformations



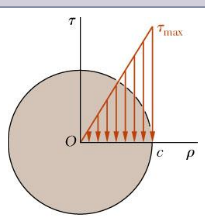
- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.
- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for
  remain because a
  circular shaft is axisymmetric.
- Cross-sections of
  when
  subjected to torsion.

Shearing Strain

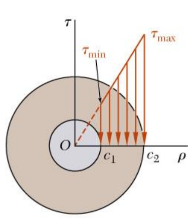


- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.
- Since the ends of the element remain planar,
- It follows that
- Shear strain is proportional to twist and radius

Stresses in Elastic Range



(a)



(b)

- Multiplying the previous equation by the shear modulus,

  From Hooke's Law, , so

  The shearing stress varies linearly with the radial position in the section.
- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,
- The results are known as the *elastic torsion formulas*,

Stresses in Elastic Range

EXAMPLE 3.01

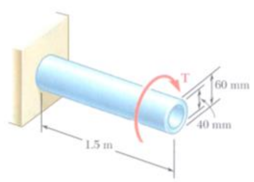


Fig. 3.15

A hollow cylindrical steel shaft is 1.5 m long and has inner and outer diameters respectively equal to 40 and 60 mm (Fig. 3.15). (*a*) What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed 120 MPa? (*b*) What is the corresponding minimum value of the shearing stress in the shaft?

(a) Largest Permissible Torque. The largest torque **T** that can be applied to the shaft is the torque for which $\tau_{max} = 120$ MPa. Since this value is less than the yield strength for steel, we can use Eq. (3.9). Solving this equation for T, we have

$$T = \frac{J\tau_{max}}{c} \qquad (3.12)$$

Recalling that the polar moment of inertia J of the cross section is given by Eq. (3.11), where $c_1 = \frac{1}{2}(40\text{ mm}) = 0.02$ m and $c_2 = \frac{1}{2}(60\text{ mm}) = 0.03$ m, we write

$$J = \tfrac{1}{2}\pi(c_2^4 - c_1^4) = \tfrac{1}{2}\pi(0.03^4 - 0.02^4) = 1.021 \times 10^{-6}\text{ m}^4$$

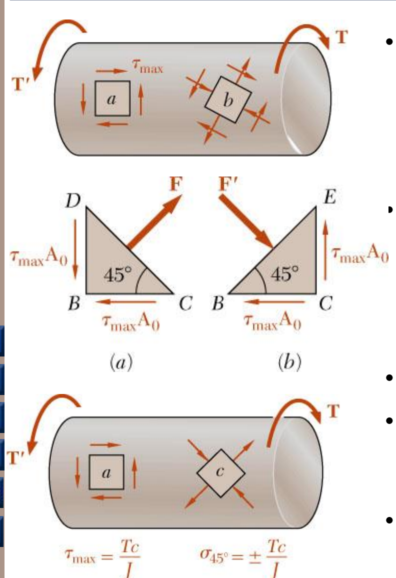
Substituting for J and τ_{max} into (3.12), and letting $c = c_2 = 0.03$ m, we have

$$T = \frac{J\tau_{max}}{c} = \frac{(1.021 \times 10^{-6}\text{ m}^4)(120 \times 10^6\text{ Pa})}{0.03\text{ m}} = 4.08\text{ kN}\cdot\text{m}$$

(b) Minimum Shearing Stress. The minimum value of the shearing stress occurs on the inner surface of the shaft. It is obtained from Eq. (3.7), which expresses that τ_{min} and τ_{max} are respectively proportional to c_1 and c_2:

$$\tau_{min} = \frac{c_1}{c_2}\tau_{max} = \frac{0.02\text{ m}}{0.03\text{ m}}(120\text{ MPa}) = 80\text{ MPa}$$

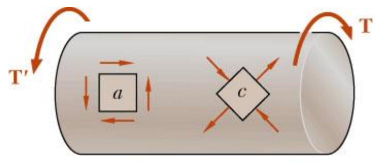
Normal Stresses



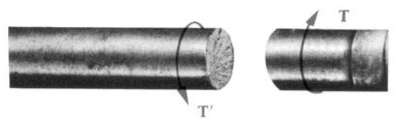
- Elements with faces

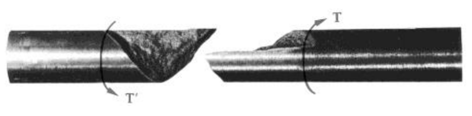
  Normal stresses, shearing stresses or a combination of both may be found for other orientations.
- Consider an element at 45° to the shaft axis,
- Element a is in pure shear.
- Element c is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that

Torsional Failure Modes



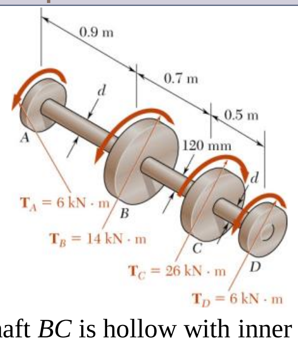
$$\tau_{max} = \frac{Tc}{J} \qquad \sigma_{45^\circ} = \pm\frac{Tc}{J}$$





- Ductile materials
  Brittle materials are
- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a
- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e.,

Sample Problem 3.1



Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

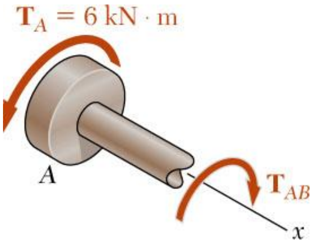
SOLUTION:

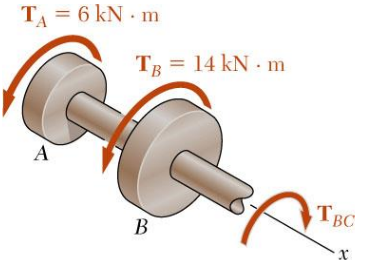
- Cut sections through shafts *AB* and *BC* and perform static equilibrium analyses to find torque loadings.
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

Sample Problem 3.1

SOLUTION:

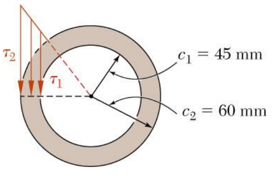
- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings.





Sample Problem 3.1

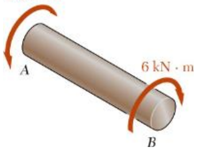
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.



$\tau_{max} = 86.2\,\text{MPa}$

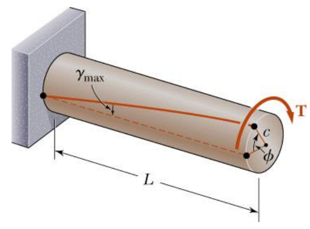
$\tau_{min} = 64.7\,\text{MPa}$

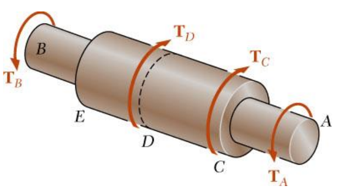
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.



$d = 2c = 77.8\,\text{mm}$

Angle of Twist in Elastic Range





- Recall that the angle of twist and maximum shearing strain are related,
- In the elastic range, the shearing strain and shear are related by Hooke's Law,
- Equating the expressions for shearing strain and solving for the angle of twist,
- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

Angle of Twist in Elastic Range

EXAMPLE 3.02

What torque should be applied to the end of the shaft of Example 3.01 to produce a twist of 2°? Use the value $G = 77$ GPa for the modulus of rigidity of steel.

Solving Eq. (3.16) for T, we write

$$T = \frac{JG}{L}\phi$$

Substituting the given values

$$G = 77 \times 10^9 \text{ Pa} \qquad L = 1.5 \text{ m}$$

$$\phi = 2°\left(\frac{2\pi \text{ rad}}{360°}\right) = 34.9 \times 10^{-3} \text{ rad}$$

and recalling from Example 3.01 that, for the given cross section,

$$J = 1.021 \times 10^{-6} \text{ m}^4$$

we have

$$T = \frac{JG}{L}\phi = \frac{(1.021 \times 10^{-6} \text{ m}^4)(77 \times 10^9 \text{ Pa})}{1.5 \text{ m}}(34.9 \times 10^{-3} \text{ rad})$$

$$T = 1.829 \times 10^3 \text{ N} \cdot \text{m} = 1.829 \text{ kN} \cdot \text{m}$$

Angle of Twist in Elastic Range

EXAMPLE 3.03

What angle of twist will create a shearing stress of 70 MPa on the inner surface of the hollow steel shaft of Examples 3.01 and 3.02?

The method of attack for solving this problem that first comes to mind is to use Eq. (3.10) to find the torque T corresponding to the given value of τ, and Eq. (3.16) to determine the angle of twist ϕ corresponding to the value of T just found.

A more direct solution, however, may be used. From Hooke's law, we first compute the shearing strain on the inner surface of the shaft:

$$\gamma_{\min} = \frac{\tau_{\min}}{G} = \frac{70 \times 10^6 \text{ Pa}}{77 \times 10^9 \text{ Pa}} = 909 \times 10^{-6}$$

Recalling Eq. (3.2), which was obtained by expressing the length of arc AA' in Fig. 3.13*c* in terms of both γ and ϕ, we have

$$\phi = \frac{L\gamma_{\min}}{c_1} = \frac{1500 \text{ mm}}{20 \text{ mm}}(909 \times 10^{-6}) = 68.2 \times 10^{-3} \text{ rad}$$

To obtain the angle of twist in degrees, we write

$$\phi = (68.2 \times 10^{-3} \text{ rad})\left(\frac{360°}{2\pi \text{ rad}}\right) = 3.91°$$

Angle of Twist in Elastic Range

EXAMPLE 3.04

For the assembly of Fig. 3.23, knowing that $r_A = 2r_B$, determine the angle of rotation of end E of shaft BE when the torque $\mathbf{T}$ is applied at E.

We first determine the torque $\mathbf{T}_{AD}$ exerted on shaft AD. Observing that equal and opposite forces $\mathbf{F}$ and $\mathbf{F}'$ are applied on the two gears at C (Fig. 3.24), and recalling that $r_A = 2r_B$, we conclude that the torque exerted on shaft AD is twice as large as the torque exerted on shaft BE; thus, $T_{AD} = 2T$.

Since the end D of shaft AD is fixed, the angle of rotation ϕ_A of gear A is equal to the angle of twist of the shaft and is obtained by writing

$$\phi_A = \frac{T_{AD}L}{JG} = \frac{2TL}{JG}$$

Observing that the arcs CC' and CC'' in Fig. 3.23*b* must be equal, we write $r_A\phi_A = r_B\phi_B$ and obtain

$$\phi_B = (r_A/r_B)\phi_A = 2\phi_A$$

We have, therefore,

$$\phi_B = 2\phi_A = \frac{4TL}{JG}$$

Considering now shaft BE, we recall that the angle of twist of the shaft is equal to the angle $\phi_{E/B}$ through which end E rotates with respect to end B. We have

$$\phi_{E/B} = \frac{T_{BE}L}{JG} = \frac{TL}{JG}$$

The angle of rotation of end E is obtained by writing

$$\phi_E = \phi_B + \phi_{E/B}$$
$$= \frac{4TL}{JG} + \frac{TL}{JG} = \frac{5TL}{JG}$$

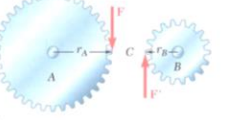
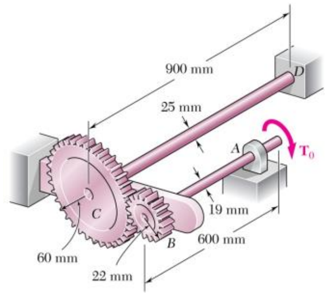


Fig. 3.24

Sample Problem 3.4



Two solid steel shafts are connected by gears. Knowing that for each shaft G = 77 GPa and that the allowable shearing stress is 55 MPa, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

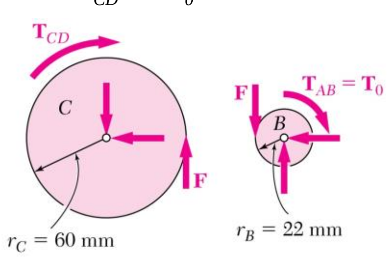
SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0 .
- Apply a kinematic analysis to relate the angular rotations of the gears.
- Find the maximum allowable torque on each shaft – choose the smallest.
- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.

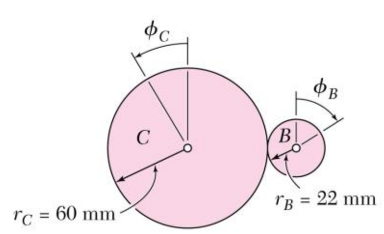
Sample Problem 3.4

SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0 .

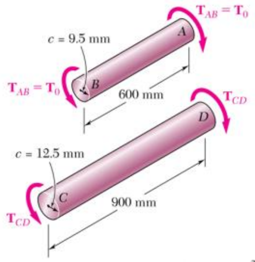


- Apply a kinematic analysis to relate the angular rotations of the gears.



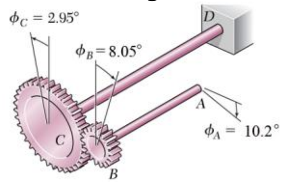
Sample Problem 3.4

- Find the T_0 for the maximum allowable torque on each shaft – choose the smallest.



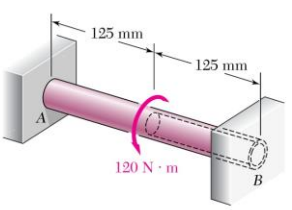
$$\boxed{T_0 = 61.8\,\text{Nm}}$$

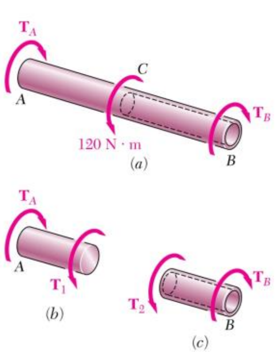
- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.



$$\boxed{\phi_A = 10.2^{\circ}}$$

Statically Indeterminate Shafts





- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.
- From a free-body analysis of the shaft,

$$T_A + T_B = 120\,\text{N}\cdot\text{m}$$

which is not sufficient to find the end torques. The problem is statically indeterminate.

- Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

- Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 120\,\text{N}\cdot\text{m}$$

Statically Indeterminate Shafts

EXAMPLE 3.05 A circular shaft *AB* consists of a 250-mm-long, 22-mm-diameter steel cylinder, in which a 125-mm-long, 16-mm-diameter cavity has been drilled from end *B*. The shaft is attached to fixed supports at both ends, and a 120-N · m torque is applied at its midsection (Fig. 3.25). Determine the torque exerted on the shaft by each of the supports.

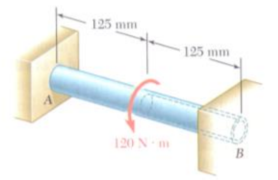


Fig. 3.25

Drawing the free-body diagram of the shaft and denoting by $\mathbf{T}_A$ and $\mathbf{T}_B$ the torques exerted by the supports (Fig. 3.26*a*), we obtain the equilibrium equation

$$T_A + T_B = 120 \text{ N} \cdot \text{m}$$

Since this equation is not sufficient to determine the two unknown torques $\mathbf{T}_A$ and $\mathbf{T}_B$, the shaft is statically indeterminate.

However, $\mathbf{T}_A$ and $\mathbf{T}_B$ can be determined if we observe that the total angle of twist of shaft *AB* must be zero, since both of its ends are restrained. Denoting by ϕ_1 and ϕ_2, respectively, the angles of twist of portions *AC* and *CB*, we write

$$\phi = \phi_1 + \phi_2 = 0$$

Statically Indeterminate Shafts

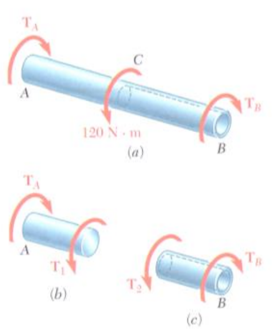


Fig. 3.26

From the free-body diagram of a small portion of shaft including end *A* (Fig. 3.26*b*), we note that the internal torque T_1 in *AC* is equal to T_A; from the free-body diagram of a small portion of shaft including end *B* (Fig. 3.26*c*), we note that the internal torque T_2 in *CB* is equal to T_B. Recalling Eq. (3.16) and observing that portions *AC* and *CB* of the shaft are twisted in opposite senses, we write

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0$$

Solving for T_B, we have

$$T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

Substituting the numerical data

$$L_1 = L_2 = 125 \text{ mm}$$
$$J_1 = \tfrac{1}{2}\pi(0.011 \text{ m})^4 = 230 \times 10^{-6} \text{ m}^4$$
$$J_2 = \tfrac{1}{2}\pi[(0.011 \text{ m})^4 - (0.008 \text{ m})^4] = 165.6 \times 10^{-6} \text{ m}^4$$

we obtain

$$T_B = 0.72\, T_A$$

Substituting this expression into the original equilibrium equation, we write

$$1.72\, T_A = 120 \text{ N} \cdot \text{m}$$

$$T_A = 69.8 \text{ N} \cdot \text{m} \qquad T_B = 50.2 \text{ N} \cdot \text{m}$$

Design of Transmission Shafts

- Principal transmission shaft performance specifications are:
  - power
  - speed
- Designer must select shaft material and cross-section to meet performance specifications without exceeding allowable shearing stress.
- Determine torque applied to shaft at specified power and speed,
- Find shaft cross-section which will not exceed the maximum allowable shearing stress,

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Design of Transmission Shafts

EXAMPLE 3.06

What size of shaft should be used for the rotor of a 3.7-kW motor operating at 3600 rpm if the shearing stress is not to exceed 60 MPa in the shaft?

We first express the power of the motor in N · m/s and its frequency in cycles per second (or hertzes).

$$P = 3.7 \text{ kW} = 3700 \text{ N} \cdot \text{m/s}$$

$$f = (3600 \text{ rpm})\frac{1 \text{ Hz}}{60 \text{ rpm}} = 60 \text{ Hz} = 60 \text{ s}^{-1}$$

The torque exerted on the shaft is given by Eq. (3.21):

$$T = \frac{P}{2\pi f} = \frac{3700 \text{ N} \cdot \text{m/s}}{2\pi\,(60 \text{ s}^{-1})} = 9.815 \text{ N} \cdot \text{m}$$

Substituting for T and τ_{max} into Eq. (3.22), we write

$$\frac{J}{c} = \frac{T}{\tau_{max}} = \frac{9.815 \text{ N} \cdot \text{m}}{60 \text{ MPa}} = 163.58 \text{ mm}^3$$

But $J/c = \frac{1}{2}\pi c^3$ for a solid shaft. We have, therefore,

$$\frac{1}{2}\pi c^3 = 163.58 \text{ mm}^3$$
$$c = 4.705 \text{ mm}$$
$$d = 2c = 9.41 \text{ mm}$$

A 10-mm shaft should be used.

Design of Transmission Shafts

EXAMPLE 3.07 A shaft consisting of a steel tube of 50-mm outer diameter is to transmit 100 kW of power while rotating at a frequency of 20 Hz. Determine the tube thickness that should be used if the shearing stress is not to exceed 60 MPa.

The torque exerted on the shaft is given by Eq. (3.21):

$$T = \frac{P}{2\pi f} = \frac{100 \times 10^3 \text{ W}}{2\pi \text{ (20 Hz)}} = 795.8 \text{ N} \cdot \text{m}$$

From Eq. (3.22) we conclude that the parameter J/c_2 must be at least equal to

$$\frac{J}{c_2} = \frac{T}{\tau_{\max}} = \frac{795.8 \text{ N} \cdot \text{m}}{60 \times 10^6 \text{ N/m}^2} = 13.26 \times 10^{-6} \text{ m}^3 \qquad (3.23)$$

But, from Eq. (3.10) we have

$$\frac{J}{c_2} = \frac{\pi}{2c_2}(c_2^4 - c_1^4) = \frac{\pi}{0.050}[(0.025)^4 - c_1^4] \qquad (3.24)$$

Equating the right-hand members of Eqs. (3.23) and (3.24), we obtain:

$$(0.025)^4 - c_1^4 = \frac{0.050}{\pi}(13.26 \times 10^{-6})$$

$$c_1^4 = 390.6 \times 10^{-9} - 211.0 \times 10^{-9} = 179.6 \times 10^{-9} \text{ m}^4$$

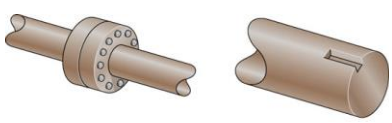
$$c_1 = 20.6 \times 10^{-3} \text{ m} = 20.6 \text{ mm}$$

The corresponding tube thickness is

$$c_2 - c_1 = 25 \text{ mm} - 20.6 \text{ mm} = 4.4 \text{ mm}$$

A tube thickness of 5 mm should be used.

Stress Concentrations



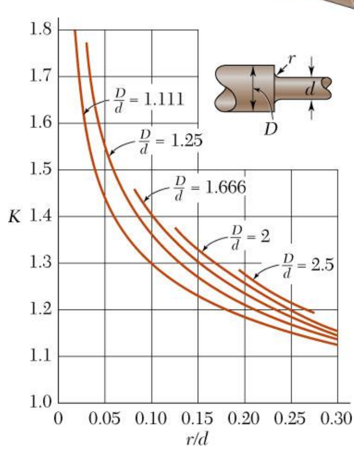


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.

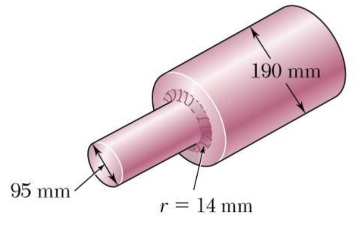
- The derivation of the torsion formula,

  assumed a circular shaft with uniform cross-section loaded through rigid end plates.

- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and

- Experimental or numerically determined concentration factors are applied as

Sample Problem 3.6



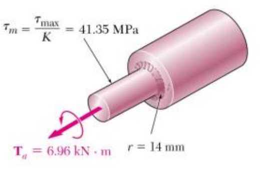
A stepped shaft is to rotate at 900 rpm as it transmits power, and its allowable shearing stress is 50 MPa. For the design shown, determine (*a*) the maximum transmittance power, (*b*) the power change when the fillet radius (*r*) is increased to 24 mm. Consider the stress concentration effect in each case.

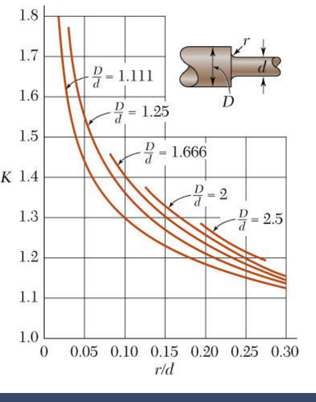
SOLUTION:

- Determine stress concentration factor (K) according to the given dimensions.
- Calculate the maximum torque corresponding to the allowable shearing stress
- Calculate the resulting transmission power for the given angular frequency.
- Repeat the same procedure for the increase corner radius value.

Sample Problem 3.6

Initial design





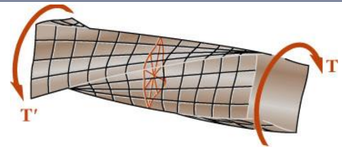
Modified design

$$r = 24\,\text{mm}, \qquad \frac{r}{d} = \frac{24}{95} = 0.253$$

$$T = \frac{J}{c}\frac{\tau_{max}}{K} = \frac{\pi c^3}{2}\frac{\tau_{max}}{K} = 7.708\,kN \cdot m$$

$$P' = 2\pi f T = 2\pi(15)(7.708) = 726.5 kW$$

Torsion of Noncircular Members



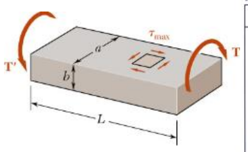
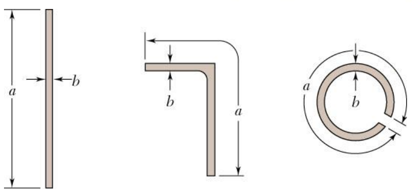


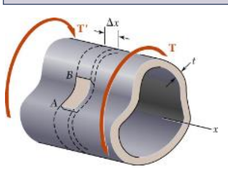
TABLE 3.1. Coefficients for Rectangular Bars in Torsion

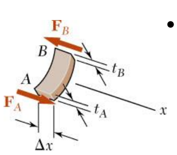
a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

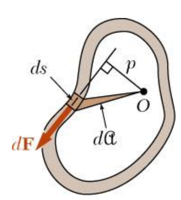


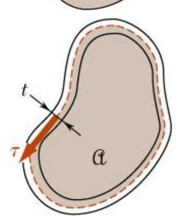
- Previous torsion formulas are valid for axisymmetric or circular shafts
- Planar cross-sections of noncircular shafts and stress and strain distribution
- For uniform rectangular cross-sections,
- At large values of *a/b*, the maximum shear stress and angle of twist for other open sections are the same as a rectangular bar.

Thin-Walled Hollow Shafts







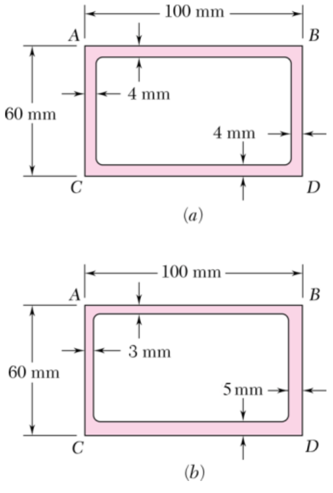


- Summing forces in the x-direction on *AB*,

shear stress varies inversely with thickness

- Compute the shaft torque from the integral of the moments due to shear stress
- Angle of twist (from Chapter 11)

Example 3.10



Extruded aluminum tubing with a rectangular cross-section has a torque loading of 2.7 kNm. Determine the shearing stress in each of the four walls with (a) uniform wall thickness of 4 mm and wall thicknesses of (b) 3 mm on *AB* and *CD* and 5 mm on *CD* and *BD*.

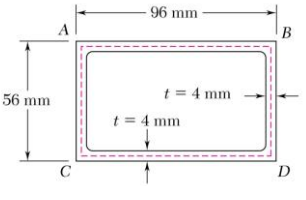
SOLUTION:

- Determine the shear flow through the tubing walls.
- Find the corresponding shearing stress with each wall thickness .

Example 3.10

SOLUTION:

- Determine the shear flow through the tubing walls.



- Find the corresponding shearing stress with each wall thickness.

With a uniform wall thickness,

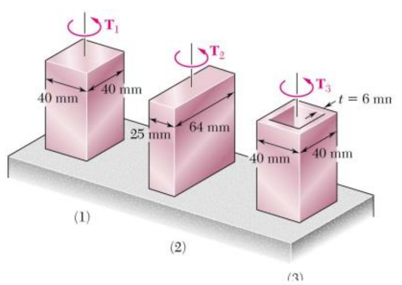
$$\boxed{\tau = 62.8\,\text{MPa}}$$

With a variable wall thickness

$$\boxed{\tau_{AB} = \tau_{BC} = 83.7\,\text{MPa}}$$

$$\boxed{\tau_{BC} = \tau_{CD} = 50.2\,\text{MPa}}$$

Sample Problem 3.9



For the allowable shearing stress of 40 MPa, determine the largest torque that may be applied to each of the given brass bars and tube. (Note: two solid bars have the same cross-sectional area; the square bar and tube have the same outside dimensions)

SOLUTION:

- Determine stress concentration factor (K) according to the given dimensions.
- Calculate the maximum torque corresponding to the allowable shearing stress
- Calculate the resulting transmission power for the given angular frequency.
- Repeat the same procedure for the increase corner radius value.

Sample Problem 3.9

1) Square bar

$$\frac{a}{b}=\frac{40}{40}=1.0 \quad \Rightarrow c_1 = 0.208$$

2) Rectangular bar

$$\frac{a}{b}=\frac{64}{25} \quad \Rightarrow c_1 = 2.56 \text{ (from linear interpolation)}$$

3) Square tube

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3.0	0.267	0.263
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5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333