Sixth GE SI Edition

CHAPTER

4

# MECHANICS OF MATERIALS

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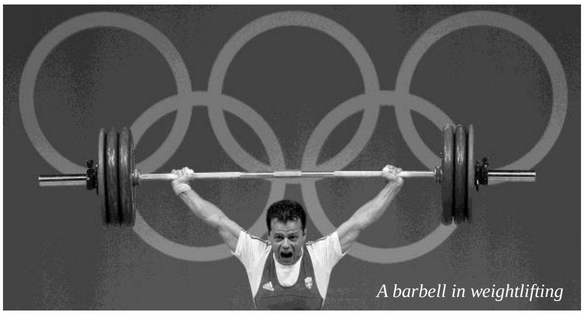
## Pure Bending

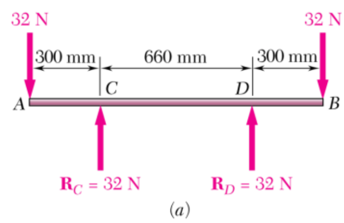




## Contents

## Pure Bending


A barbell in weightlifting




- Prismatic members (각기둥) subjected to

  acting in the same longitudinal plane
- Used in the design of many machine and structural components (eg.beams and girders)



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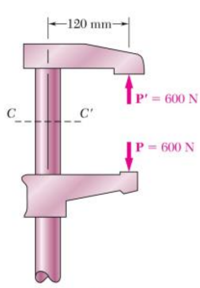
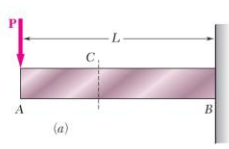
## Other Loading Types inducing Bending



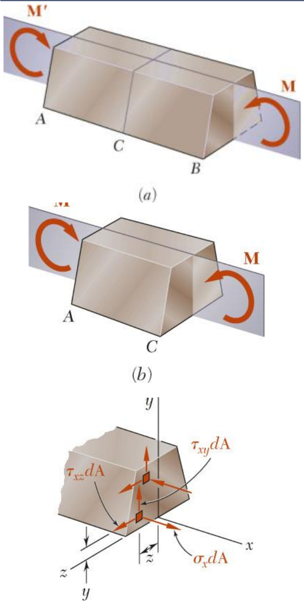




- *Eccentric Loading*: Axial loading which does not pass through section centroid produces
- *Transverse Loading*: Concentrated or distributed transverse load produces
- *Principle of Superposition*: The normal stress due to pure bending may be

  to find the complete state of stress.

End

## Symmetric Member in Pure Bending

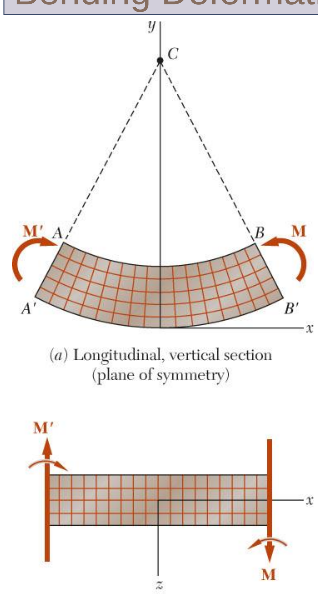



- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is
  and zero
  about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.






## Bending Deformations




(a) Longitudinal, vertical section (plane of symmetry)

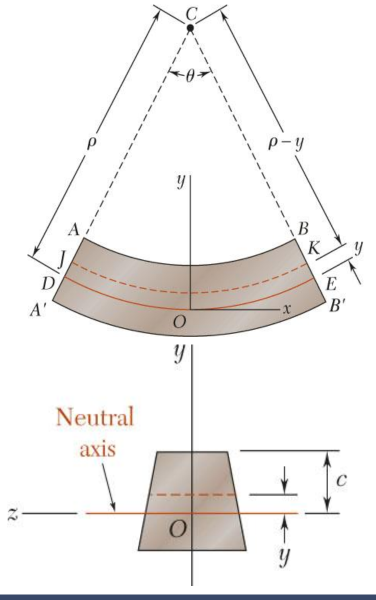
(b) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly
- cross-sectional plane passes through arc center and
- length of top decreases and length of bottom increases

  must exist that is parallel to the upper and lower surfaces and for which
- stresses and strains are
  above the neutral plane and
  below the neutral plane

## Strain Due to Bending




Consider a beam segment of length $L$.

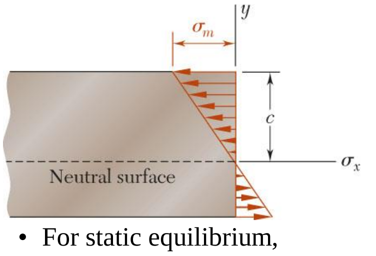
After deformation, the length of the neutral surface remains $L$. At other sections,






## Stress Due to Bending

- For a linearly elastic material,




- For static equilibrium,

- For static equilibrium,

First moment with respect to neutral plane is zero. Therefore,

# Stress Due to Bending

**EXAMPLE 4.01**


**Fig. 4.14**

A steel bar of 20 × 60-mm rectangular cross section is subjected to two equal and opposite couples acting in the vertical plane of symmetry of the bar (Fig. 4.14). Determine the value of the bending moment $M$ that causes the bar to yield. Assume $\sigma_Y = 250$ MPa.

Since the neutral axis must pass through the centroid $C$ of the cross section, we have $c = 30$ mm (Fig. 4.15). On the other hand, the centroidal moment of inertia of the rectangular cross section is

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(20\text{ mm})(60\text{ mm})^3 = 360 \times 10^3\text{ mm}^4$$

Solving Eq. (4.15) for $M$, and substituting the above data, we have

$$M = \frac{I}{c}\sigma_m = \frac{360 \times 10^{-9}\text{ m}^4}{0.03\text{ m}}(250\text{ MPa})$$

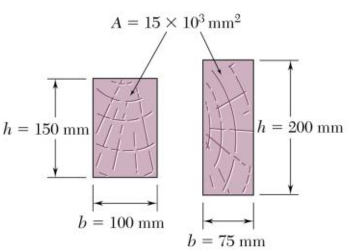
$$M = 3\text{ kN} \cdot \text{m}$$

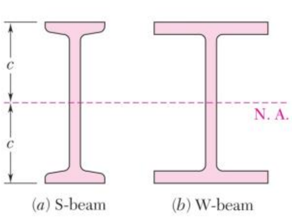






# Beam Section Properties




(a) S-beam (b) W-beam

- The maximum normal stress due to bending,

  A beam section with a larger section modulus will have a lower maximum stress

- Consider a rectangular beam cross section,

  Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

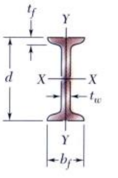
- Structural steel beams are designed to have a large section modulus.

# Properties of American Standard Shapes



**Appendix C.** Properties of Rolled-Steel Shapes
(SI Units)

**S Shapes**
(American Standard Shapes)

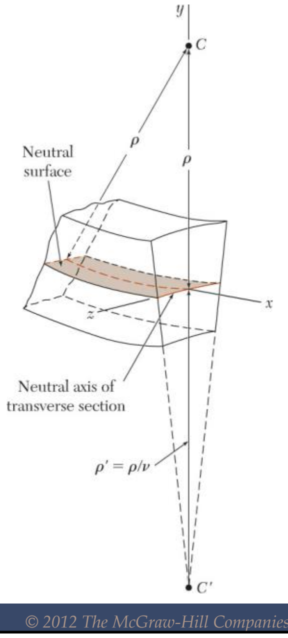



| Designation† | Area $A$, mm² | Depth $d$, mm | Flange Width $b_f$, mm | Flange Thickness $t_f$, mm | Web Thickness $t_w$, mm | Axis X-X $I_x$ $10^6$ mm⁴ | Axis X-X $S_x$ $10^3$ mm³ | Axis X-X $r_x$ mm | Axis Y-Y $I_y$ $10^6$ mm⁴ | Axis Y-Y $S_y$ $10^3$ mm³ | Axis Y-Y $r_y$ mm |
|---|---|---|---|---|---|---|---|---|---|---|---|
| S610 × 180 | 22900 | 622 | 204 | 27.7 | 20.3 | 1320 | 4240 | 240 | 34.9 | 341 | 39.0 |
| 158 | 20100 | 622 | 200 | 27.7 | 15.7 | 1230 | 3950 | 247 | 32.5 | 321 | 39.9 |
| 149 | 19000 | 610 | 184 | 22.1 | 18.9 | 995 | 3260 | 229 | 20.2 | 215 | 32.3 |
| 134 | 17100 | 610 | 181 | 22.1 | 15.9 | 938 | 3080 | 234 | 19.0 | 206 | 33.0 |
| 119 | 15200 | 610 | 178 | 22.1 | 12.7 | 878 | 2880 | 240 | 17.9 | 198 | 34.0 |
| S510 × 143 | 18200 | 516 | 183 | 23.4 | 20.3 | 700 | 2710 | 196 | 21.3 | 228 | 33.9 |
| 128 | 16400 | 516 | 179 | 23.4 | 16.8 | 658 | 2550 | 200 | 19.7 | 216 | 34.4 |
| 112 | 14200 | 508 | 162 | 20.2 | 16.1 | 530 | 2090 | 193 | 12.6 | 152 | 29.5 |
| 98.3 | 12500 | 508 | 159 | 20.2 | 12.8 | 495 | 1950 | 199 | 11.8 | 145 | 30.4 |
| S460 × 104 | 13300 | 457 | 159 | 17.6 | 18.1 | 385 | 1685 | 170 | 10.4 | 127 | 27.5 |
| 81.4 | 10400 | 457 | 152 | 17.6 | 11.7 | 333 | 1460 | 179 | 8.83 | 113 | 28.8 |
| S380 × 74 | 9500 | 381 | 143 | 15.6 | 14.0 | 201 | 1060 | 145 | 6.65 | 90.8 | 26.1 |
| 64 | 8150 | 381 | 140 | 15.8 | 10.4 | 185 | 971 | 151 | 6.15 | 85.7 | 27.1 |



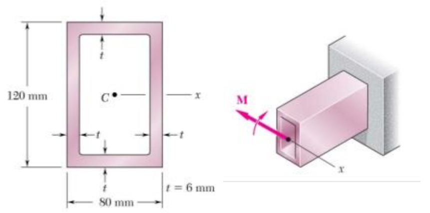



# Deformations in a Transverse Cross Section




- Deformation due to bending moment $M$ is quantified by the curvature of the neutral surface
- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,
- Expansion above the neutral surface and contraction below it cause an in-plane curvature,

## Sample Problem 4.1




A hollow rectangular tube is extruded from an aluminum alloy for which SY = 275 MPa, SU = 415 MPa, and $E$ = 73 GPa. Neglecting the effects of fillets, determine (a) the bending moment M for which the safety factor will be 3.0, (b) the corresponding radius of curvature of the tube.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y} A}{\sum A} \qquad I_{x'} = \sum\left(\bar{I} + A d^2\right)$$

- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

- Calculate the curvature

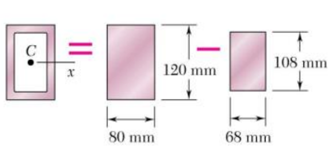
$$\frac{1}{\rho} = \frac{M}{EI}$$

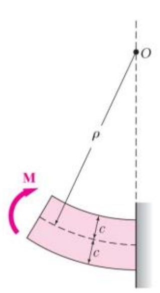






## Sample Problem 4.1




SOLUTION:

Moment of inertia.

$$I = \tfrac{1}{12} 80 \times 120^3 - \tfrac{1}{12} 68 \times 108^3 = 4.382 \times 10^6 \, \text{mm}^4$$

Allowable stress (F.S. = 3.0)

$$\sigma_{all} = \frac{\sigma_U}{F.S.} = \frac{415\,\text{MPa}}{3.0} = 138.33\,\text{MPa}$$

Bending moment (with c = 60 mm)

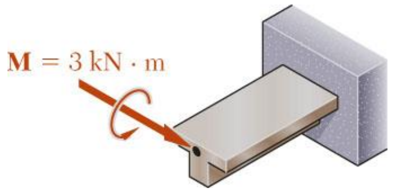
$$M = \frac{I}{c}\sigma_{all} = \frac{4.382 \times 10^{-6}\,\text{m}^4}{0.06\,\text{m}}(138.33\,\text{MPa}) = 10.1\,\text{kN}\cdot\text{m}$$

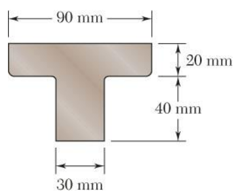
Radius of curvature

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{10.1\,\text{kN}\cdot\text{m}}{(73\text{GPa})(4.382 \times 10^{-6}\,\text{m}^4)} = 0.0316\,\text{m}^{-1}$$

$$\therefore \quad \rho = 31.7\,\text{m}$$

## Sample Problem 4.2




A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing $E$ = 165 GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

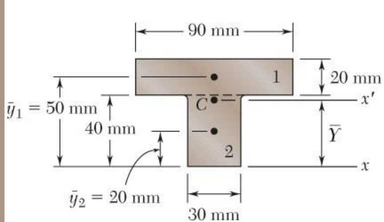
SOLUTION:

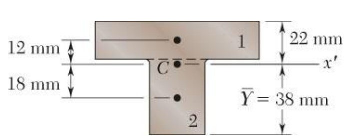
- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.
- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.
- Calculate the curvature






## Sample Problem 4.2

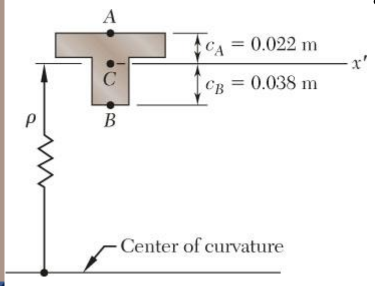



SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

## Sample Problem 4.2




- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_A = +76.0\ \text{MPa}$$

$$\sigma_B = -131.3\ \text{MPa}$$

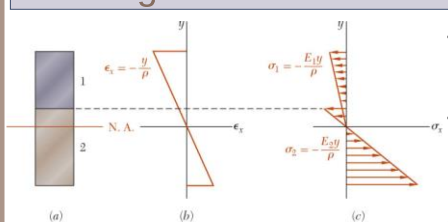
- Calculate the curvature

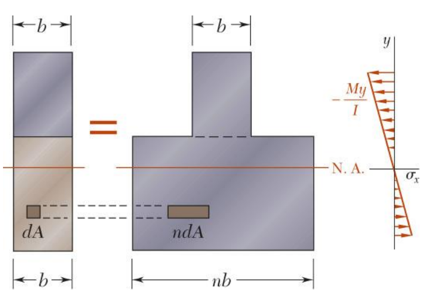
$$\frac{1}{\rho} = 20.95 \times 10^{-3}\ \text{m}^{-1}$$
$$\rho = 47.7\ \text{m}$$






## Bending of Members Made of Several Materials




$$\sigma_x = -\frac{My}{I}$$

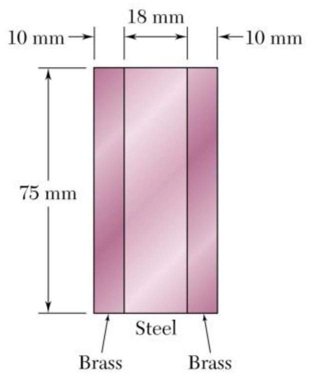
$$\sigma_1 = \sigma_x \qquad \sigma_2 = n\sigma_x$$

- Consider a composite beam formed from two materials with $E_1$ and $E_2$.
- Normal strain varies linearly.
- Piecewise linear normal stress variation.

Neutral axis does not pass through section centroid of composite section.

- Elemental forces on the section are
- Define a transformed section such that

## Example 4.03




Bar is made from bonded pieces of steel ($E_s$ = 200 GPa) and brass ($E_b$ = 100 GPa). Determine the maximum stress in the steel and brass when a moment of 4.5 KNm is applied.

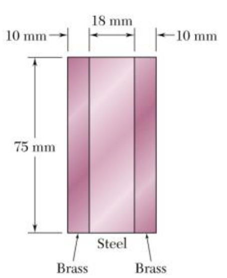
SOLUTION:

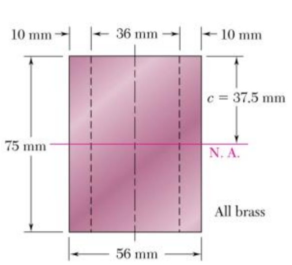
- Transform the bar to an equivalent cross section made entirely of brass
- Evaluate the cross sectional properties of the transformed section
- Calculate the maximum stress in the transformed section. This is the correct maximum stress for the brass pieces of the bar.
- Determine the maximum stress in the steel portion of the bar by multiplying the maximum stress for the transformed section by the ratio of the moduli of elasticity.






## Example 4.03



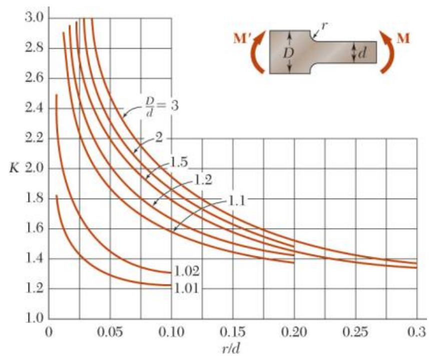



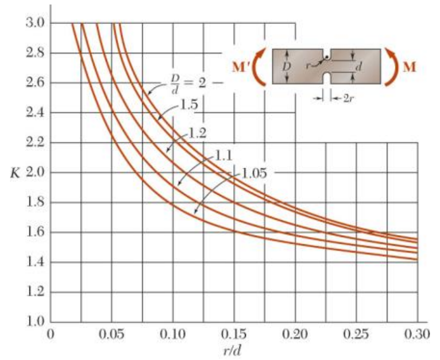

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.
- Evaluate the transformed cross sectional properties
- Calculate the maximum stresses

$$(\sigma_b)_{max} = 85.7\,\text{MPa}$$
$$(\sigma_s)_{max} = 171.4\,\text{MPa}$$

# Stress Concentrations




Stress concentrations may occur:

- in the vicinity of points

- in the vicinity of






# Stress Concentrations

**EXAMPLE 4.04**

Grooves 10 mm deep are to be cut in a steel bar which is 60 mm wide and 9 mm thick (Fig. 4.29). Determine the smallest allowable width of the grooves if the stress in the bar is not to exceed 150 MPa when the bending moment is equal to 180 N · m.

We note from Fig. 4.29*a* that

$$d = 60\text{ mm} - 2(10\text{ mm}) = 40\text{ mm}$$
$$c = \tfrac{1}{2}d = 20\text{ mm} \qquad b = 9\text{ mm}$$

The moment of inertia of the critical cross section about its neutral axis is

$$I = \tfrac{1}{12}bd^3 = \tfrac{1}{12}(9\times10^{-3}\text{ m})(40\times10^{-3}\text{ m})^3 = 48\times10^{-9}\text{ m}^4$$

The value of the stress $Mc/I$ is thus

$$\frac{Mc}{I} = \frac{(180\text{ N}\cdot\text{m})(20\times10^{-3}\text{ m})}{48\times10^{-9}\text{ m}^4} = 75\text{ MPa}$$

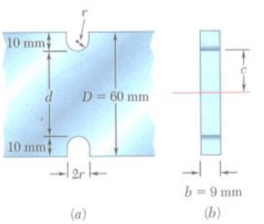



Fig. 4.29

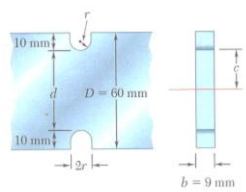
## Stress Concentrations

Substituting this value for $Mc/I$ into Eq. (4.29) and making $\sigma_m = 150$ MPa, we write

$$150 \text{ MPa} = K(75 \text{ MPa})$$
$$K = 2$$

We have, on the other hand,

$$\frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$$



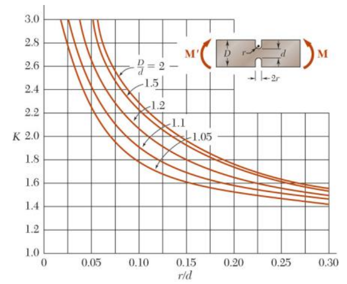

Using the curve of Fig. 4.28 corresponding to $D/d = 1.5$, we find that the value $K = 2$ corresponds to a value of $r/d$ equal to 0.13. We have, therefore,

$$\frac{r}{d} = 0.13$$

$$r = 0.13d = 0.13(40 \text{ mm}) = 5.2 \text{ mm}$$

The smallest allowable width of the grooves is thus

$$2r = 2(5.2 \text{ mm}) = 10.4 \text{ mm}$$

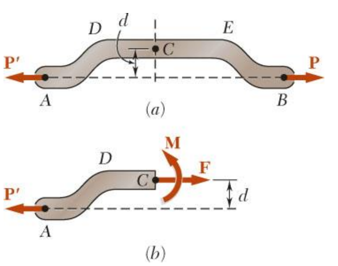








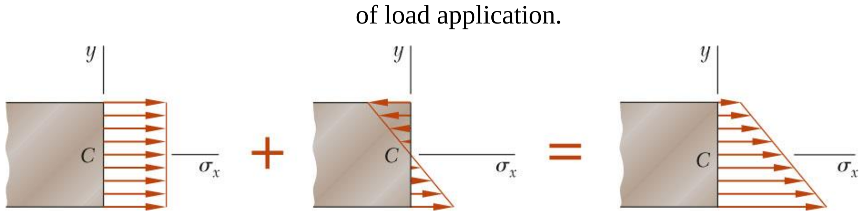
## Eccentric Axial Loading in a Plane of Symmetry



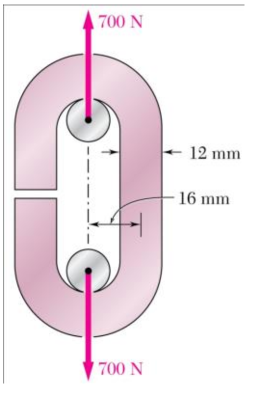

- Eccentric loading

- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and linear stress distribution due a pure bending moment

- Validity requires stresses below proportional limit, deformations have negligible effect on geometry, and stresses not evaluated near points of load application.



## Example 4.07



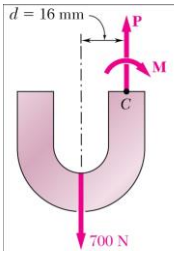

•An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For 700 N load, determine (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis

- SOLUTION:
- Find the equivalent centric load and bending moment
- Superpose the uniform stress due to the centric load and the linear stress due to the bending moment.
- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.
- Find the neutral axis by determining the location where the normal stress is zero.

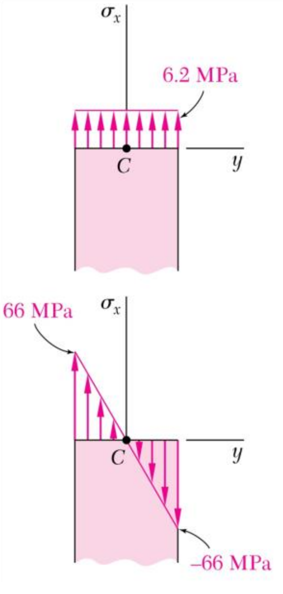





## Example 4.07

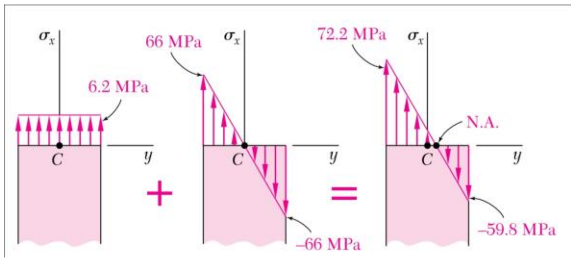



- Equivalent centric load and bending moment




- Normal stress due to a centric load
- Normal stress due to bending moment

## Example 4.07




- Maximum tensile and compressive stresses

$$\sigma_t = 72.2\,\text{MPa}$$

$$\sigma_c = -59.8\,\text{MPa}$$

- Neutral axis location

$$y_0 = 0.56\,\text{mm}$$

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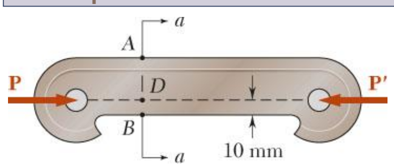
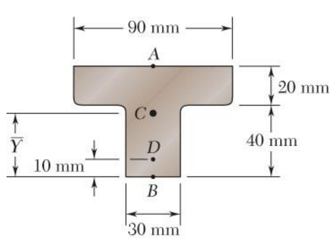
## Sample Problem 4.8




Section a–a

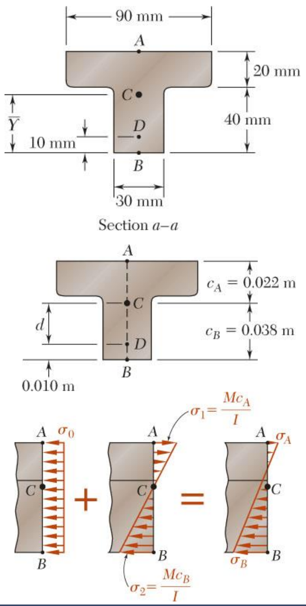
•From Sample Problem 4.2,

$$A = 3\times10^{-3}\,\text{m}^2$$
$$\bar{Y} = 0.038\,\text{m}$$
$$I = 868\times10^{-9}\,\text{m}^4$$

•The largest allowable stresses for the cast iron link are 30 MPa in tension and 120 MPa in compression. Determine the largest force $P$ which can be applied to the link.

- SOLUTION:
- Determine equivalent centric load and bending moment.
- Superpose the stress due to a centric load and the stress due to bending.
- Evaluate the critical loads for the allowable tensile and compressive stresses.
- The largest allowable load is the smallest of the two critical loads.

## Sample Problem 4.8

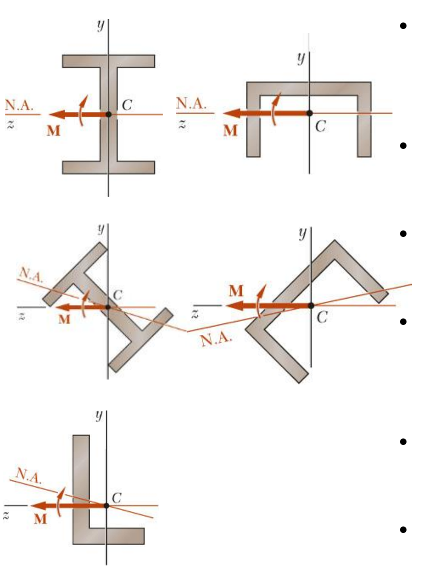



- Determine equivalent centric and bending loads.
- Superpose stresses due to centric and bending loads
- Evaluate critical loads for allowable stresses.
- The largest allowable load $P = 77.0$ kN



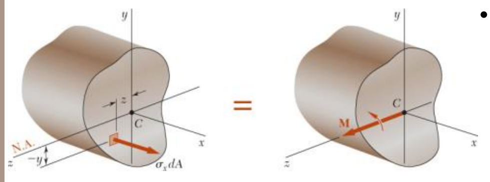



## Unsymmetric Bending




- Analysis of pure bending has been limited to members subjected to bending couples acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- The neutral axis of the cross section coincides with the axis of the couple.
- Will now consider situations in which the bending couples do not act in a plane of symmetry.
- Cannot assume that the member will bend in the plane of the couples.
- In general, the neutral axis of the section will not coincide with the axis of the couple.

## Unsymmetric Bending




Wish to determine the conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple as shown.

- The resultant force and moment from the distribution of elementary forces in the section must satisfy

- neutral axis passes through centroid

- defines stress distribution

- couple vector must be directed along a principal centroidal axis

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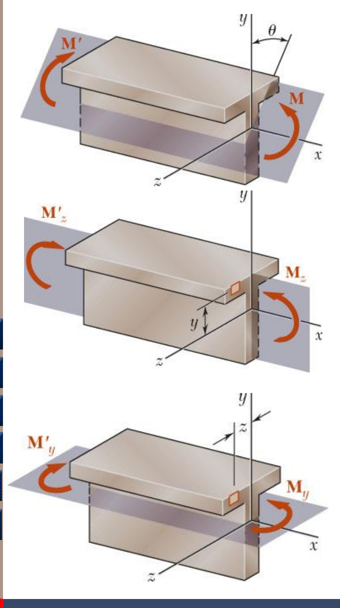
## Unsymmetric Bending

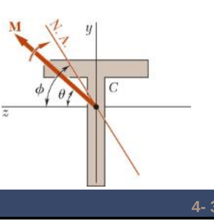



Superposition is applied to determine stresses in the most general case of unsymmetric bending.

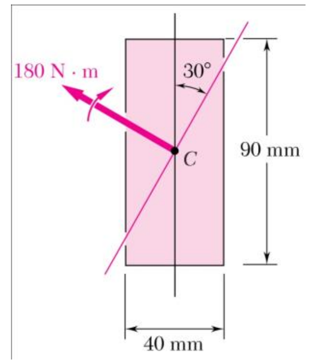
- Resolve the couple vector into components along the principle centroidal axes.

- Superpose the component stress distributions

- Along the neutral axis,



## Example 4.08




A 180 Nm couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.

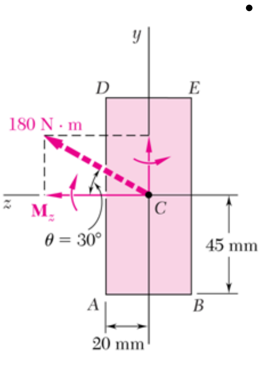
SOLUTION:

- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.
- Combine the stresses from the component stress distributions.
- Determine the angle of the neutral axis.






## Example 4.08




- Resolve the couple vector into components and calculate the corresponding maximum stresses.

$$M_z = (180\,\text{Nm})\cos 30 = 155.9\,\text{Nm}$$
$$M_y = (180\,\text{Nm})\sin 30 = 90\,\text{Nm}$$
$$I_z = \tfrac{1}{12}(0.04\,\text{m})(0.09\,\text{m})^3 = 2.43\times10^{-6}\,\text{m}^4$$
$$I_y = \tfrac{1}{12}(0.09\,\text{m})(0.04\,\text{m})^3 = 0.48\times10^{-6}\,\text{m}^4$$

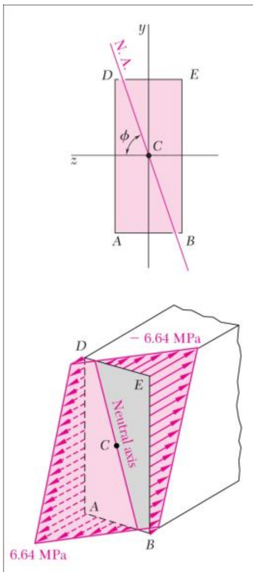
The largest tensile stress due to $M_z$ occurs along $AB$

The largest tensile stress due to $M_z$ occurs along $AD$

- The largest tensile stress due to the combined loading occurs at *A*.

$$\sigma_{max} = 6.64\,\text{MPa}$$

## Example 4.08




- Determine the angle of the neutral axis.

$\phi = 71^{\circ}$





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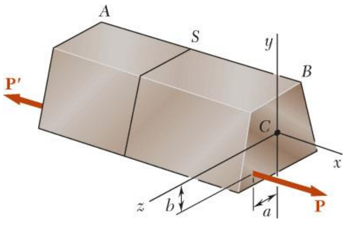
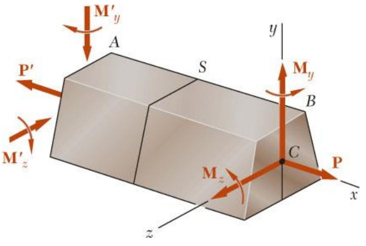
## General Case of Eccentric Axial Loading




- Consider a straight member subject to equal and opposite eccentric forces.
- The eccentric force is equivalent to the system of a centric force and two couples.

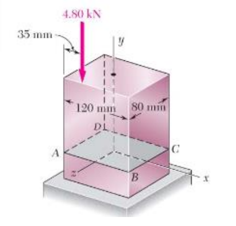



- By the principle of superposition, the combined stress distribution is
- If the neutral axis lies on the section, it may be found from

# General Case of Eccentric Axial Loading

**EXAMPLE 4.09**

A vertical 4.80-kN load is applied as shown on a wooden post of rectangular cross section, 80 by 120 mm (Fig. 4.65). (*a*) Determine the stress at points *A*, *B*, *C*, and *D*. (*b*) Locate the neutral axis of the cross section.

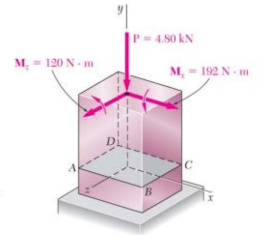



**(a) Stresses.** The given eccentric load is replaced by an equivalent system consisting of a centric load **P** and two couples $\mathbf{M}_x$ and $\mathbf{M}_z$ represented by vectors directed along the principal centroidal axes of the section (Fig. 4.66). We have

$$M_x = (4.80\text{ kN})(40\text{ mm}) = 192\text{ N}\cdot\text{m}$$
$$M_z = (4.80\text{ kN})(60\text{ mm} - 35\text{ mm}) = 120\text{ N}\cdot\text{m}$$

We also compute the area and the centroidal moments of inertia of the cross section:

$$A = (0.080\text{ m})(0.120\text{ m}) = 9.60 \times 10^{-3}\text{ m}^2$$
$$I_x = \tfrac{1}{12}(0.120\text{ m})(0.080\text{ m})^3 = 5.12 \times 10^{-6}\text{ m}^4$$
$$I_z = \tfrac{1}{12}(0.080\text{ m})(0.120\text{ m})^3 = 11.52 \times 10^{-6}\text{ m}^4$$




The stress $\sigma_0$ due to the centric load **P** is negative and uniform across the section. We have

$$\sigma_0 = \frac{P}{A} = \frac{-4.80\text{ kN}}{9.60 \times 10^{-3}\text{ m}^2} = -0.5\text{ MPa}$$









# General Case of Eccentric Axial Loading

The stresses due to the bending couples $\mathbf{M}_x$ and $\mathbf{M}_z$ are linearly distributed across the section, with maximum values equal, respectively, to

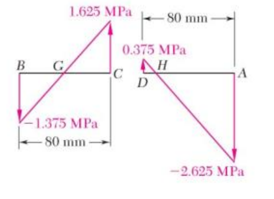
$$\sigma_1 = \frac{M_x z_{max}}{I_x} = \frac{(192\text{ N}\cdot\text{m})(40\text{ mm})}{5.12 \times 10^{-6}\text{ m}^4} = 1.5\text{ MPa}$$
$$\sigma_2 = \frac{M_z x_{max}}{I_z} = \frac{(120\text{ N}\cdot\text{m})(60\text{ mm})}{11.52 \times 10^{-6}\text{ m}^4} = 0.625\text{ MPa}$$

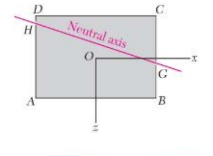
The stresses at the corners of the section are

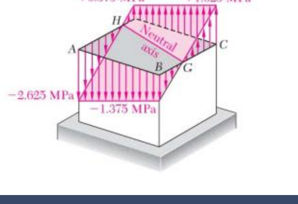
$$\sigma_y = \sigma_0 \pm \sigma_1 \pm \sigma_2$$

where the signs must be determined from Fig. 4.66. Noting that the stresses due to $\mathbf{M}_x$ are positive at *C* and *D*, and negative at *A* and *B*, and that the stresses due to $\mathbf{M}_z$ are positive at *B* and *C*, and negative at *A* and *D*, we obtain

$$\sigma_A = -0.5 - 1.5 - 0.625 = -2.625\text{ MPa}$$
$$\sigma_B = -0.5 - 1.5 + 0.625 = -1.375\text{ MPa}$$
$$\sigma_C = -0.5 + 1.5 + 0.625 = +1.625\text{ MPa}$$
$$\sigma_D = -0.5 + 1.5 - 0.625 = +0.375\text{ MPa}$$




**(b) Neutral Axis.** We note that the stress will be zero at a point *G* between *B* and *C*, and at a point *H* between *D* and *A* (Fig. 4.67). Since the stress distribution is linear, we write

$$\frac{BG}{80\text{ mm}} = \frac{1.375}{1.625 + 1.375} \qquad BG = 36.7\text{ mm}$$
$$\frac{HA}{80\text{ mm}} = \frac{2.625}{2.625 + 0.375} \qquad HA = 70\text{ mm}$$