Sixth GE SI Edition

CHAPTER

7

MECHANICS OF MATERIALS

Ferdinand P. Beer
E. Russell Johnston, Jr.
John T. DeWolf
David F. Mazurek

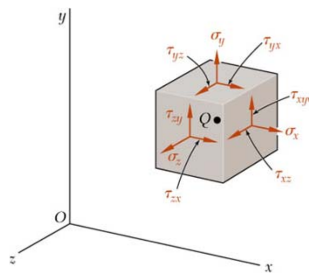
Lecture Notes:
J. Walt Oler
Texas Tech University

Modified by
Prof. Keun Park
SeoulTech, Korea

Transformations of Stress and Strain

Contents

Introduction

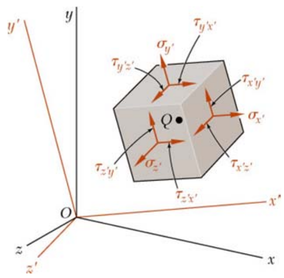


- The most general state of stress at a point may be represented by 6 components,

  $\sigma_x, \sigma_y, \sigma_z$ normal stresses

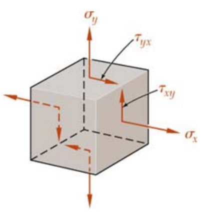
  $\tau_{xy}, \tau_{yz}, \tau_{zx}$ shearing stresses

  $(\text{Note: } \tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz})$

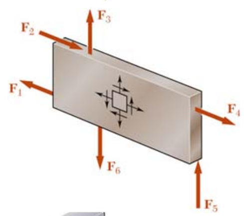


- The first part of the chapter is concerned with how the components of stress are transformed under a rotation of the coordinate axes.
- The second part of the chapter is devoted to a similar analysis of the transformation of the components of strain (Skip).

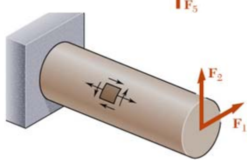
Introduction



- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by

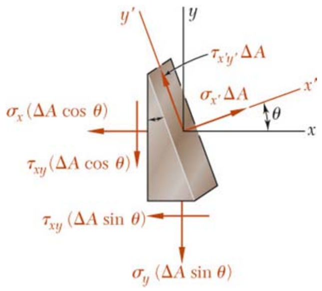


- State of plane stress occurs in   subjected to forces acting in the midplane of the plate. (i.e.



- State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.

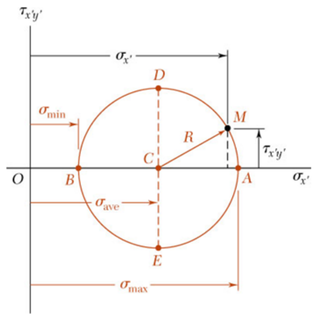
Transformation of Plane Stress

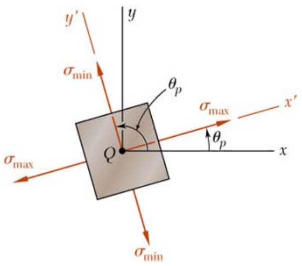


- Consider the conditions for equilibrium of a prismatic element with faces perpendicular to the *x*, *y*, and *x'* axes.

The equations may be rewritten to yield

Principal Stresses

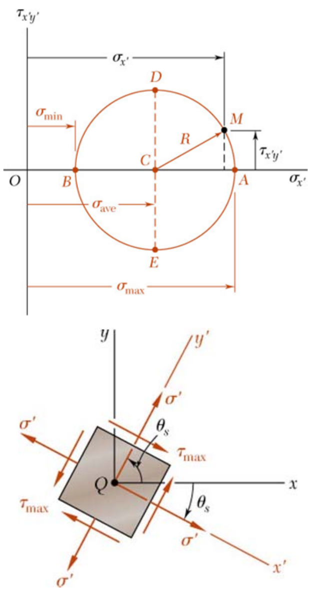




- The previous equations are combined to yield parametric equations for a circle,

- *Principal stresses* occur on the *principal planes of stress* with zero shearing stresses.

Maximum Shearing Stress



Maximum shearing stress occurs for $\sigma_{x'} = \sigma_{ave}$

Note : defines two angles separated by 90° and offset from θ_p by 45°

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

Example 7.01

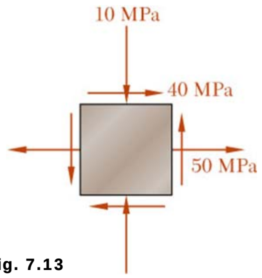


Fig. 7.13

For the state of plane stress shown, determine
(a) the principal planes,
(b) the principal stresses,
(c) the maximum shearing stress and the corresponding normal stress.

SOLUTION:

- Find the element orientation for the principal stresses from
- Determine the principal stresses from
- Calculate the maximum shearing stress with

Example 7.01

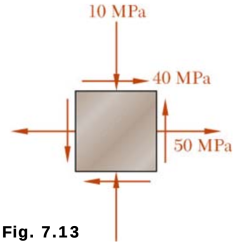


Fig. 7.13

SOLUTION:

- Find the element orientation for the principal stresses from

$$\theta_p = 26.6°, 116.6°$$

- Determine the principal stresses from

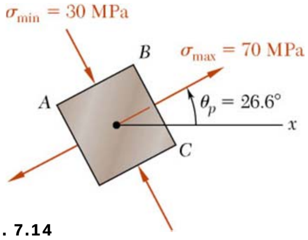


Fig. 7.14

$$\sigma_{max} = 70\,\text{MPa}$$
$$\sigma_{min} = -30\,\text{MPa}$$

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Example 7.01

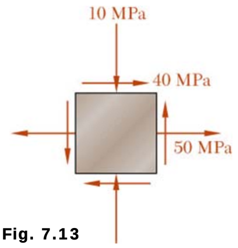


Fig. 7.13

$\sigma_x = +50\,\text{MPa}$ $\tau_{xy} = +40\,\text{MPa}$

$\sigma_x = -10\,\text{MPa}$

- Calculate the maximum shearing stress with

$$\tau_{max} = 50\,\text{MPa}$$

$$\theta_s = -18.4°, 71.6°$$

- The corresponding normal stress is

$$\sigma' = 20\,\text{MPa}$$

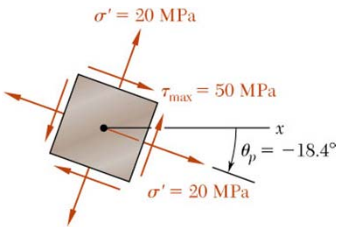
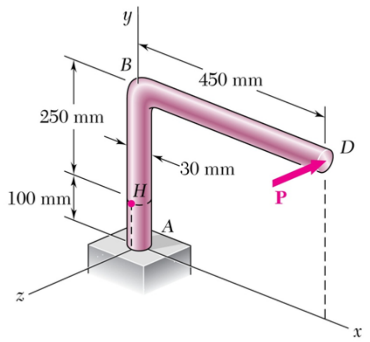


Fig. 7.16

Sample Problem 7.1

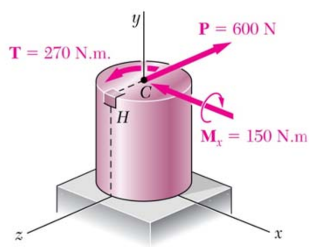


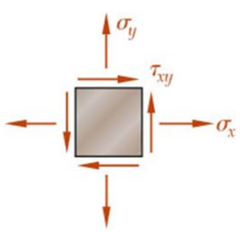
SOLUTION:

- Determine an equivalent force-couple system at the center of the transverse section passing through *H*.
- Evaluate the normal and shearing stresses at *H*.
- Determine the principal planes and calculate the principal stresses.

A single horizontal force *P* of 600 N magnitude is applied to end D of lever *ABD*. Determine (a) the normal and shearing stresses on an element at point *H* having sides parallel to the *x* and *y* axes, (b) the principal planes and principal stresses at the point *H*.

Sample Problem 7.1



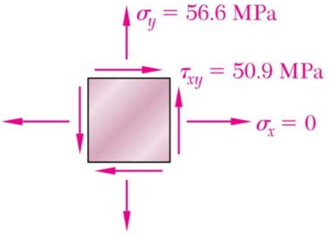


SOLUTION:

- Determine an equivalent force-couple system at the center of the transverse section passing through *H*.
- Evaluate the normal and shearing stresses at *H*.

$$\sigma_x = 0 \quad \sigma_y = +56.6\,\text{MPa} \quad \tau_y = +50.9\,\text{MPa}$$

Sample Problem 7.1



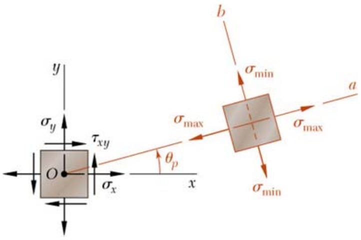
- Determine the principal planes and calculate the principal stresses.

$$\theta_p = -30.5°, 59.5°$$

$$\sigma_{max} = +86.5\,\text{MPa}$$
$$\sigma_{min} = -29.9\,\text{MPa}$$

Mohr's Circle for Plane Stress

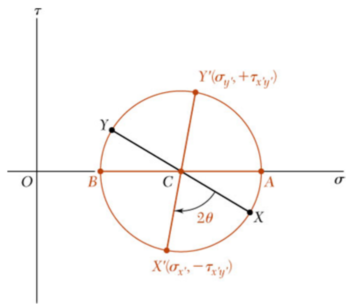
- With the physical significance of Mohr's circle for plane stress established, it may be applied with simple geometric considerations. Critical values are estimated graphically or calculated.
- For a known state of plane stress $\sigma_x, \sigma_y, \tau_{xy}$ plot the points *X* and *Y* and construct the circle centered at *C*.
- The principal stresses are obtained at *A* and *B*.

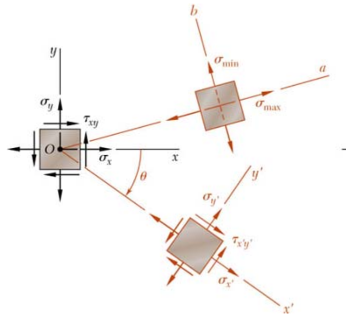


The direction of rotation of *Ox* to *Oa* is the same as *CX* to *CA* *(half in magnitude)* .

Mohr’s Circle for Plane Stress

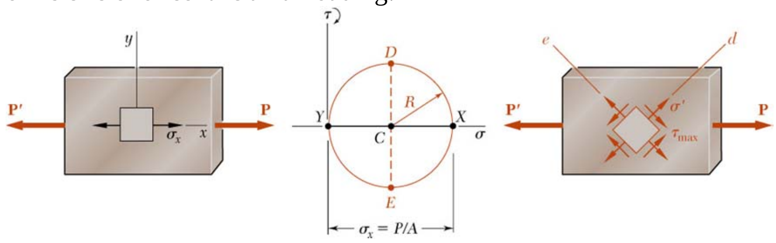
- With Mohr’s circle uniquely defined, the state of stress at other axes orientations may be depicted.
- For the state of stress at an angle θ with respect to the xy axes, construct a new diameter $X'Y'$ at an angle 2θ with respect to XY.
- Normal and shear stresses are obtained from the coordinates $X'Y'$.



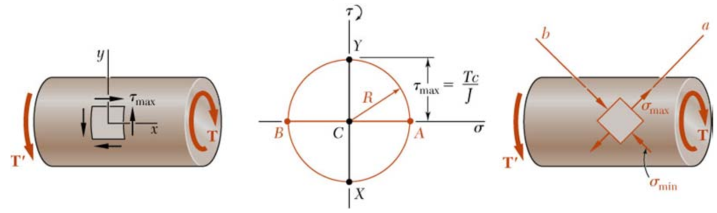


Mohr’s Circle for Plane Stress

- Mohr’s circle for centric axial loading:



- Mohr’s circle for torsional loading:



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Example 7.02

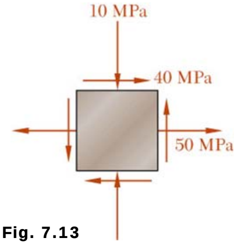

Fig. 7.13

For the state of plane stress shown, (a) construct Mohr's circle, determine (b) the principal planes, (c) the principal stresses, (d) the maximum shearing stress and the corresponding normal stress.

SOLUTION:

- Construction of Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{(50)+(-10)}{2} = 20\,\text{MPa}$$

$$CF = 50 - 20 = 30\,\text{MPa} \qquad FX = 40\,\text{MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50\,\text{MPa}$$

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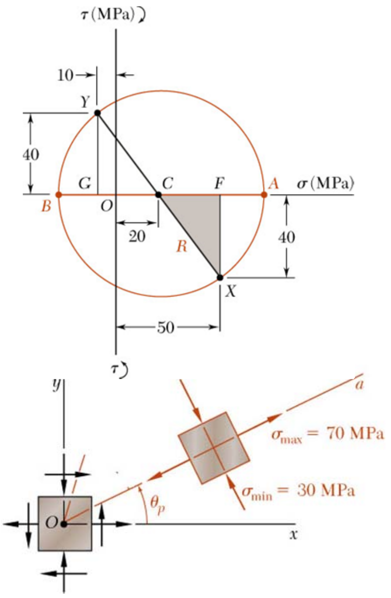
Example 7.02



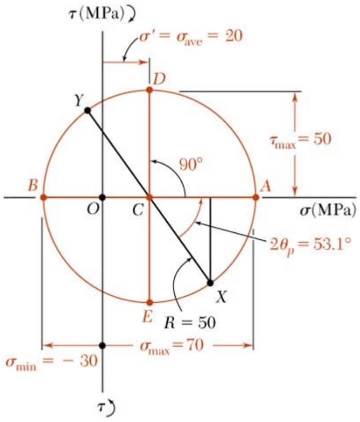
- Principal planes and stresses

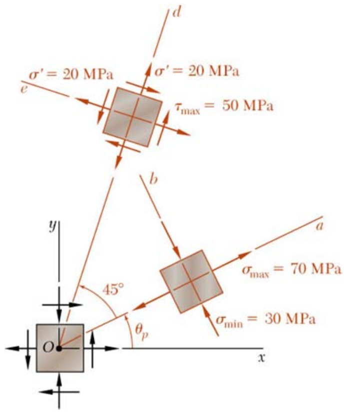
$$\sigma_{max} = 70\,\text{MPa}$$

$$\sigma_{min} = -30\,\text{MPa}$$

$$\theta_p = 26.6°$$

Example 7.02





- Maximum shear stress

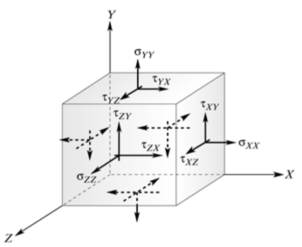
$\theta_s = 71.6°$ $\tau_{max} = 50\ \text{MPa}$ $\sigma' = 20\ \text{MPa}$

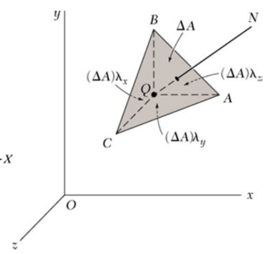
General State of Stress

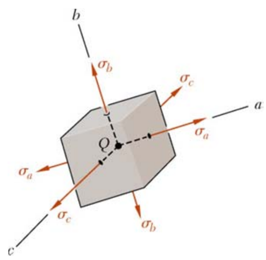
- Consider the general 3D state of stress at a point and the transformation of stress from element rotation - State of stress at Q defined by:
- Consider tetrahedron with face perpendicular to the line QN with direction cosines:
- The requirement $\sum F_n = 0$ leads to,

$$\sigma_n = \sigma_x \lambda_x^2 + \sigma_y \lambda_y^2 + \sigma_z \lambda_z^2 + 2\tau_{xy}\lambda_x\lambda_y + 2\tau_{yz}\lambda_y\lambda_z + 2\tau_{zx}\lambda_z\lambda_x$$

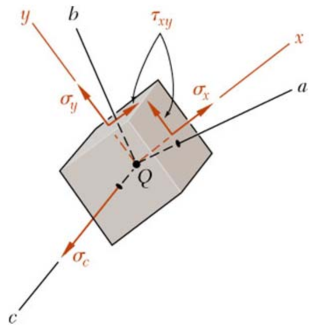
- Form of equation guarantees that an element orientation can be found such that







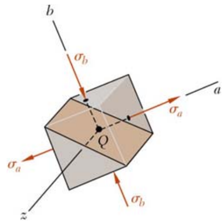
Application of Mohr's Circle to the Three-Dimensional Analysis of Stress



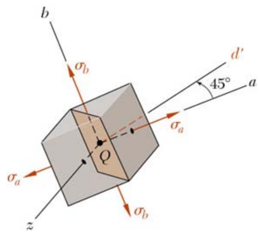
- Transformation of stress for an element rotated around a principal axis may be represented by Mohr's circle.
- Points *A*, *B*, and *C* represent the principal stresses on the principal planes (shearing stress is zero)
- The three circles represent the normal and shearing stresses for rotation around each principal axis.
- Radius of the largest circle yields the maximum shearing stress.

Application of Mohr's Circle to the Three-Dimensional Analysis of Stress

- In the case of plane stress, the axis perpendicular to the plane of stress is a principal axis
- If the points *A* and *B* (representing the principal planes) are of the origin, then
  a) the corresponding principal stresses are the maximum and minimum normal stresses for the element
  b) the maximum shearing stress for the element is equal to the maximum "in-plane" shearing stress
  c) planes of maximum shearing stress are at 45° to the principal planes.



Application of Mohr's Circle to the Three-Dimensional Analysis of Stress



- If A and B are of the origin (i.e., have the same sign), then

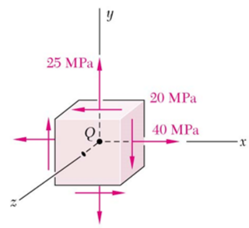
  a) the circle defining σ_{max}, σ_{min}, and τ_{max} for the element is not the circle corresponding to transformations within the plane of stress

  b) maximum shearing stress for the element is equal to half of the maximum stress

  c) planes of maximum shearing stress are at 45 degrees to the plane of stress

Example 7.03

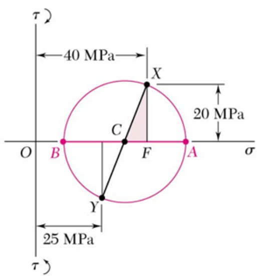
Determine (a) three principal planes and stresses and (b) the maximum shearing stress



- Mohr's circle (2-dimension)

- Principal stress (2-dimension)

- Principal stress (3-dimension)

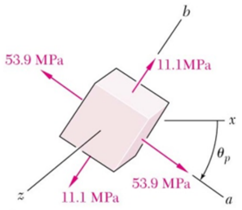


Example 7.03

- Principal direction

$$\tan 2\theta_p = \frac{FX}{CF} = \frac{20}{7.5}$$

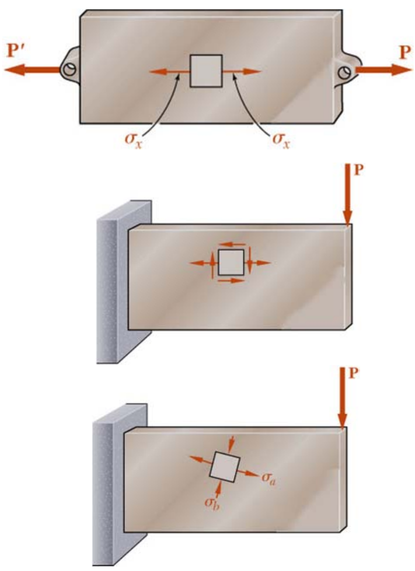
$$2\theta_p = 69.4^\circ \qquad \theta_p = 34.7^\circ$$



- The maximum shear stress

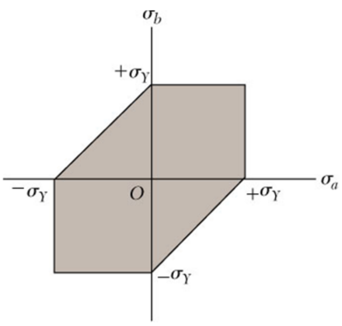
- Three-dimensional Mohr circle

Yield Criteria for Ductile Materials Under Plane Stress



- Failure of a machine component subjected to uniaxial stress is directly predicted
- Failure of a machine component subjected to plane stress cannot be directly predicted
- It is convenient to determine the and to base the failure criteria
- Failure criteria are based on the mechanism of failure. Allows comparison of the failure conditions

Yield Criteria for Ductile Materials Under Plane Stress



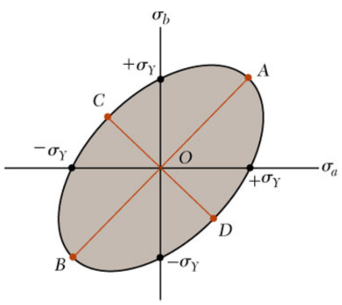
Maximum shearing stress criteria:

Structural component is safe as long as the maximum shearing stress is less than the maximum shearing stress in a tensile test specimen at yield, i.e.,

For σ_a and σ_b with the same sign,

For σ_a and σ_b with opposite signs,

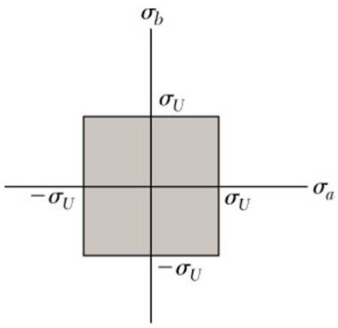
Yield Criteria for Ductile Materials Under Plane Stress



Maximum distortion energy criteria:

Structural component is safe as long as the distortion energy per unit volume is less than that occurring in a tensile test specimen at yield.

Fracture Criteria for Brittle Materials Under Plane Stress

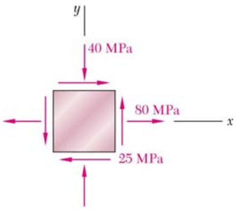


Brittle materials fail suddenly through
in a tensile test. The
failure condition is characterized

Maximum normal stress criteria:

Structural component is safe as long as the maximum normal stress is less than the ultimate strength of a tensile test specimen.

Sample Problem 7.4



The state of plane stress shown occurs at a critical point of a steel machine component, of which tensile yield strength is 250 MPa. Determine the factor of safety with respect to yield, using (a) the maximum-shearing-stress criterion, and (b) the maximum-distortion-energy criterion.

SOLUTION:

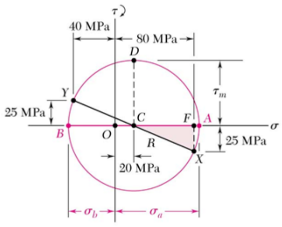
- Determine the normal and maximum shearing stresses in the spherical cap (pressure vessels)
- Determine the hoop and longitudinal stresses in the cylindrical tank
- Draw Mohr circle for the cylindrical tank
- Determine the perpendicular and parallel stress components using Mohr circle

Sample Problem 7.4

- Mohr's circle

$$\sigma_{ave} = \text{OC} = \frac{\sigma_x + \sigma_y}{2} = \frac{80-40}{2} = 20\ \text{MPa}$$

$$\tau_m = R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(60)^2 + (25)^2} = 65\ \text{MPa}$$

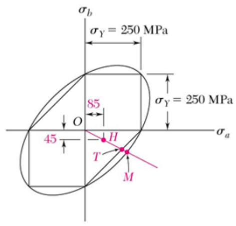


- Principal stresses

$$\sigma_a = OA = OC + CA = 20 + 65 = 85\ \text{MPa}$$

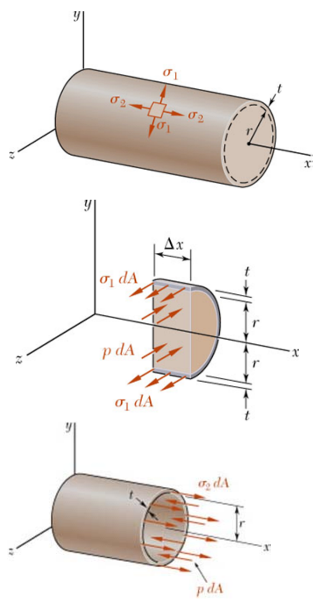
$$\sigma_b = OB = OC - BC = 20 - 65 = -45\ \text{MPa}$$

- Tresca yield criterion



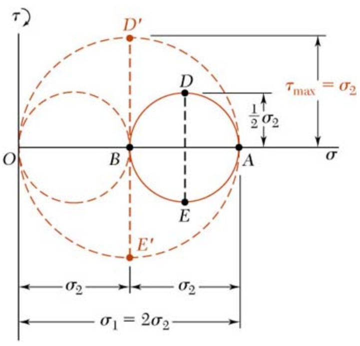
- Von-Mises yield criterion

Stresses in Thin-Walled Pressure Vessels



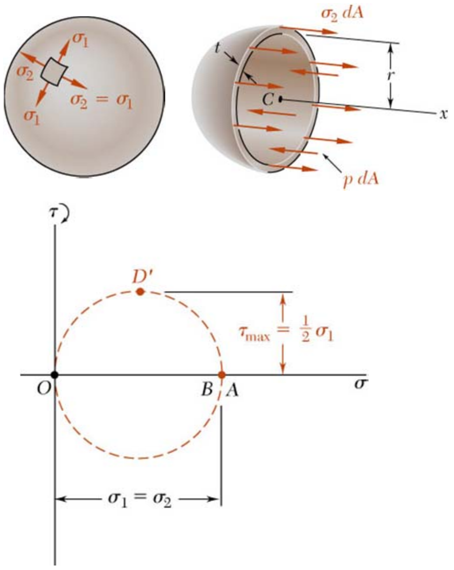
- Cylindrical vessel with principal stresses
  - σ_1 = hoop stress
  - σ_2 = longitudinal stress
- Hoop stress:
- Longitudinal stress:

Stresses in Thin-Walled Pressure Vessels



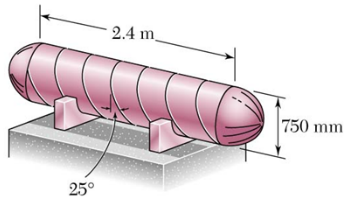
- Points *A* and *B* correspond to hoop stress, σ_1, and longitudinal stress, σ_2
- Maximum in-plane shearing stress:
- Maximum out-of-plane shearing stress corresponds to a 45° rotation of the plane stress element around a longitudinal axis

Stresses in Thin-Walled Pressure Vessels



- Spherical pressure vessel:
- Mohr's circle for in-plane transformations reduces to a point
- Maximum out-of-plane shearing stress

Sample Problem 7.5



A compressed-air tank is supported by two cradles. The tank body is fabricated from a 10-mm steel plate welded along a helix that forms an angle of 25º. Two spherical end caps have wall thicknesses of 8 mm. For an internal pressure of 1.2 MPa, determine (a) the normal and maximum shearing stresses in the spherical caps, and (b) the stresses in directions perpendicular and parallel to the helical weld.

SOLUTION:

- Determine the normal and maximum shearing stresses in the spherical cap (pressure vessels)
- Determine the hoop and longitudinal stresses in the cylindrical tank
- Draw Mohr's circle for the cylindrical tank
- Determine the perpendicular and parallel stress components using Mohr's circle

Sample Problem 7.5

- Spherical cap

$$t = 8\,\text{mm}, \quad r = 375 - 8 = 367\,\text{mm}$$

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{(1.2\,\text{MPa})(367\,\text{mm})}{2(8\,\text{mm})} = 27.5\,\text{MPa}$$

$$\tau_{max} = \tfrac{1}{2}\sigma_1 = \tfrac{1}{2}(27.5\,\text{MPa}) = 13.75\,\text{MPa}$$

- Cylindrical tank

$$t = 10\,\text{mm}, \quad r = 375 - 10 = 365\,\text{mm}$$

$$\sigma_1 = \frac{pr}{t} = \frac{(1.2\,\text{MPa})(365\,\text{mm})}{10\,\text{mm}} = 43.8\,\text{MPa} \quad \text{(hoop stress)}$$

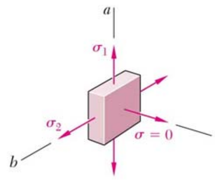
$$\sigma_2 = \tfrac{1}{2}\sigma_1 = 21.9\,\text{MPa} \quad \text{(longitudinal stress)}$$

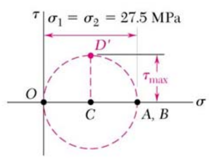
$$\sigma_{ave} = \tfrac{1}{2}(\sigma_1 + \sigma_2) = 32.85\,\text{MPa}, \quad \text{R} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 10.95\,\text{MP}$$

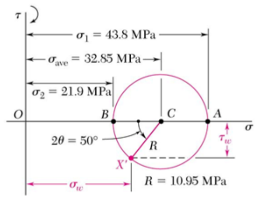
- Stress at the weld region (perpendicular/parallel)

$$\sigma_w = \sigma_{ave} - R\cos 50° = 32.85 - 10.95\cos 50° = 25.8\,\text{MPa}$$

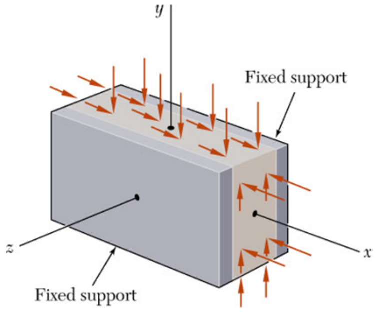
$$\tau_w = R\sin 50° = 10.95\cos 50° = 8.39\,\text{MPa}$$





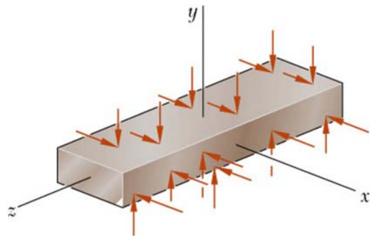


Transformation of Plane Strain



- *Plane strain* - deformations of the material take place in parallel planes and are the same in each of those planes.
- Plane strain occurs in

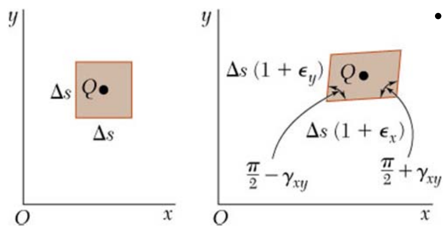
  and restrained from expanding or contracting laterally by smooth, rigid and fixed supports



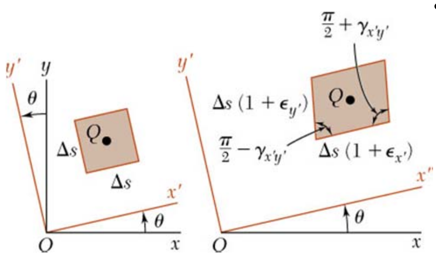
- Example: Consider

  State of plane stress exists in any transverse section not located too close to the ends of the bar.

Transformation of Plane Strain

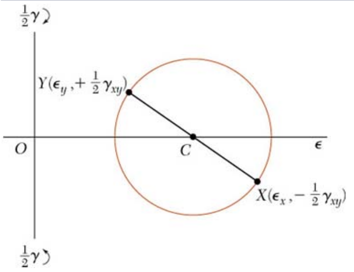


- State of strain at the point Q results in different strain components with respect to the *xy* and *x'y'* reference frames.

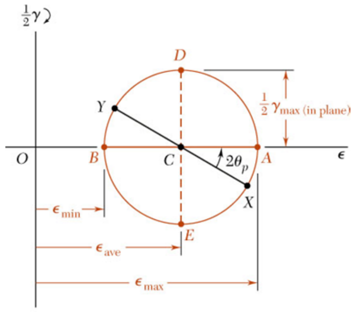


- Applying the trigonometric relations used for the transformation of stress,

Mohr's Circle for Plane Strain

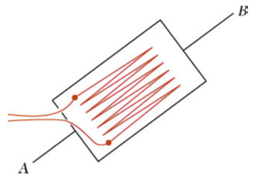


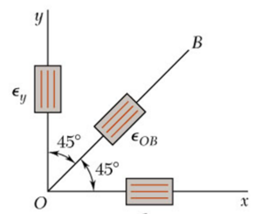
- The equations for the transformation of plane strain are of the same form as the equations for the transformation of plane stress - *Mohr's circle techniques apply*.
- Abscissa for the center *C* and radius *R* ,

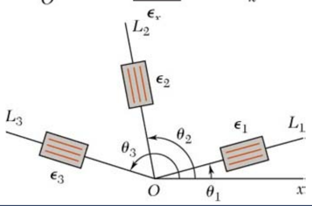


- Principal axes of strain and principal strains,
- Maximum in-plane shearing strain,

Measurements of Strain: Strain Rosette

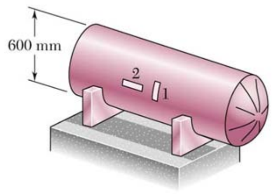






- Strain gages indicate normal strain through changes in resistance.
- With a 45° rosette, ε_x and ε_y are measured directly. γ_{xy} is obtained indirectly with,
- Normal and shearing strains may be obtained from normal strains in any three directions, (cf. 60° rosette in Sample 7.7)

Sample Problem 7.6



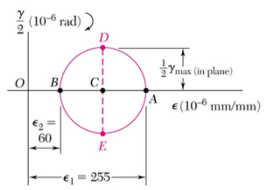
A cylindrical storage tank used to transport gas under pressure has an inner diameter of 600 mm and a wall thickness of 18 mm. Transverse and longitudinal strains are measured to 255μ and 60μ by two strain gages. Knowing that G = 77 GPa, determine (a) the gage pressure inside the tank, and (b) the principal stresses and maximum shearing stress in the tank

SOLUTION:

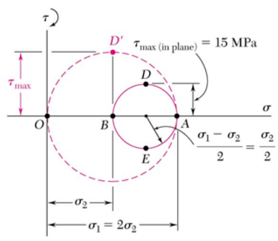
- Draw Mohr's circle for the given strain condition
- Determine the maximum shearing strain and stress
- Determine the gage pressure inside the tank
- Draw Mohr's circle for three dimensional stress
- Determine the principal stresses and the maximum shearing stress using Mohr's circle

Sample Problem 7.6

- The maximum shearing strain



- The maximum shearing stress
- Gage pressure inside the tank
- Principal stress



- The maximum shearing stress